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The crack surface anomalous scaling and its connection with the size-scale effects

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Abstract. The size-scale effects is one of the most important research topics in solid mechanics. Several theories have been proposed in order to describe the scaling of mechanical properties in fracture mechanics of quasi-brittle materials such as concrete, rock, wood and a broad class of fibrous or particulate composites. In the last two decades they were investigated by means of several techniques, including renormalisation group theory, intermediate asymptotics, dimensional analysis, statistics of extremes among the others. One of the most successful approaches is the fractal one. It is based on the assumption of a fractal-like damage localization at the mesostructural level and on the linking of mechanical properties to the fractal dimensions of the damage domains. In particular, the fractal dimension of fracture surface can be linked to the scaling properties of toughness. On the other side, recent experimental researches have shown that fracture surfaces present an anisotropic propagation in the longitudinal and transverse directions. To describe such anisotropy, it does not appear sufficient to characterize the fracture surface by a single fractal dimension, but the *anomalous scaling* (Morel et al., *Physical Review E* **58**, 6999–7005 [1998]) should be introduced. This approach has proved to be very effective in describing the R-curve behaviour (Morel et al., *International Journal of Fracture* **114**, 307–325 [2002]). Dealing with the size-scaling effects, a scaling law for both fracture toughness and tensile strength has been recently proposed. In this work, we point out some inconsistencies of the proposed approach, suggesting a more consistent way to derive the scaling laws and a correction on the scaling exponent at the larger scales. The phenomenon of scaling in notched and un-notched structures is summarized in a unified framework and the anomalous scaling is applied to the case of unnotched specimens, showing how it captures correctly only the convexity of the scaling law in a bilogarithmic plane and not the real asymptotes, thus indicating that the anomalous scaling can not be considered as a satisfactory explanation to the size-scale effects.

Key words: Anomalous scaling, Fractals, fracture, Fracture toughness, Quasi-brittle materials, Scaling, size effect, tensile strength.

1. Introduction

Scaling is a fundamental feature of almost any physical theory. In solid mechanics, the study of the so-called *size effect* is one of the most active research fields and its importance has been widely recognized during the last two decades. With size effect we mean the dependence of one or more material parameters on the size of the structure made by that material. Particularly, in quasi brittle materials such as concrete, rocks, wood and several fibrous or particulate composites, tensile strength decreases with increasing structural size, whereas fracture energy increases.

A great step ahead in comprehending this phenomenon is the introduction of the *fractality* concept, i.e. the idea that the fractal geometry of fracture surface and damage domain could be linked to the fracturing process and consequently to the size-scale effects on the material properties. Since the pioneering paper by Mandelbrot et al. (1984), the study of the fracture surfaces morphology has continuously grown and is, at the present time, a very active research field. In fact, the relation between quantities describing the geometrical structure of crack surfaces and the mechanical properties arising from fracture energetics is of fundamental importance in investigating the physical nature of fracture and in describing the scaling of material properties. Due to this reason, the development and refinement of experimental techniques for the characterization of fractal surfaces can give new tools to describe and interpret the size-scale effects.

The choice of the best method to compute the fractal dimension of a crack surface is still debated and fracture topology of the crack surfaces has been demonstrated to be much more complicated than that displayed by a simple self-affinity. The complete surface morphology of fracture surfaces in quasi-brittle materials has been reported only in the late 1990s, showing that the crack development is anisotropic, i.e. different in the crack propagation direction and along the transverse direction. To describe this anisotropy, one possibility is to introduce the so-called *anomalous scaling* (Morel et al., 1998), which is characterized, as will be shown, by three different exponents and not only by a single fractal dimension.

In the present paper, we survey some recent results obtained on this basis by Morel et al. (2002a) interpreting some experimental results of mode I fracture tests on wood specimens (Morel et al., 1998). After a brief summary on the mathematical formulation of the anomalous scaling and the description of the simple and clear procedure used by the Authors to obtain the R-curve behaviour and a very satisfactory fit of experimental data, we examine the proposed scaling laws for strength and toughness and particularly their connection with the Size Effect Law (SEL) by Bažant (1976, 1984). We point out some inconsistencies which stem from the application of the SEL hypotheses to the anomalous scaling framework. This will lead to a correction of the scaling exponent for both strength and toughness at the larger size. Eventually, we apply the anomalous scaling formalism also to the case of uncracked specimens, showing how it can capture only the concavity (or convexity) of the scaling law, but not the correct asymptotic behaviour at the smaller and larger scales. Summarizing the case of notched and unnotched specimens in a unified framework, we show that the anomalous scaling can not provide a complete explanation to the size-scale effects.

2. The Anomalous Scaling

The study of the morphology of fracture surfaces has attracted large research interests in the last few years. Observations on a broad class of different materials – quasi-brittle such as rocks (Schmittbuhl et al., 1993, 1995), wood (Engøy et al., 1994; Morel et al., 1998), concrete (Stroeven, 1991; Carpinteri and Ferro, 1994; Carpinteri et al., 1999), ceramics (Mecholsky et al., 1989; Måløy et al., 1992), or ductile such as metals (Mandelbrot et al., 1984; Bouchaud et al., 1990; Måløy et al., 1992; Imre et al., 1992) and intermetallic alloys (Bouchaud et al., 1993) – have shown that fracture surfaces display self-affine scaling properties, at least in a certain range of scales, which is in most cases very large and which strongly depends on the material microstructure.

In most studies, this scaling is characterized by a scaling exponent ζ_{loc} (sometimes called the *local roughness exponent*), which is the Hurst exponent of the self-affine surface, thus directly related to the fractal dimension D_{F} through the well-known expression $D_{\mathrm{F}} = 3 - \zeta_{\mathrm{loc}}$. This result confirms that the fracture processes are generally scale independent and implies that the macroscopic scale is related to the microscopic one (Charkaluk et al., 1998). On the other hand, there is a large scatter on experimental results, so that a number of researchers is still doubtful about the correlation of the fractal dimension of the fracture surface to any mechanical properties. Others simply affirm that the fractality of the fracture surfaces can not be the explanation of size-scale effects (Bažant, 1997a).

Despite of these objections and the related scientific controversy, the use of fractal geometry to describe cracking phenomena is now widespread and widely accepted. Moreover, there are researchers who investigate the full three-dimensional description of the fracture surfaces, evidencing a clear anisotropy in the crack development, with a diffusive behaviour in the longitudinal direction, i.e. in the crack propagation direction (Dougan et al., 2000). Others (Morel et al., 1998), studying three-dimensional maps of crack surface in wood specimens subjected to Mode I fracture, identify a type of scaling originally developed in statistical physics and obtained in some models of nonequilibrium kinetic roughening. This type of scaling, called *anomalous scaling*, distinguishes the global and local description of the surface and is characterized by three exponents. It was firstly applied with success to other physical phenomena such as Molecular Beam Epitaxy (MBE) and random diffusion (Das Sarma et al., 1994; López et al., 1997).

With reference to Figure 1, where a doubly tapered cantilever beam specimen with an initial notch is represented, if the roughness of the crack surface (or height–height correlation function) is characterized by the mean square root of height fluctuations $\Delta h(l, y)$ evaluated over a window of size l along the x direction, at a distance y from the initial notch tip, according to Morel et al. (2003) it can be written

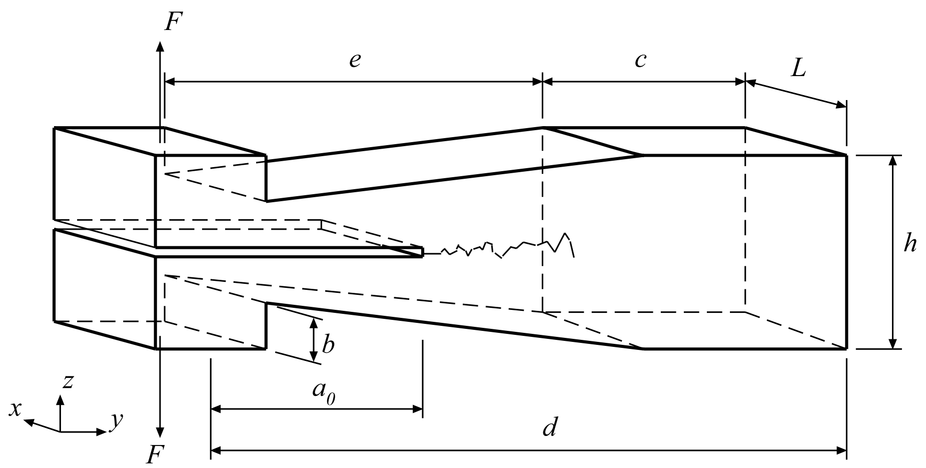

Figure 1. Doubly tapered cantilever beam specimen geometry (Morel et al., 1998).

$$\Delta h(l, y) = A \begin{cases} l^{\zeta_{\rm loc}} \xi(y)^{\zeta - \zeta_{\rm loc}}, & l \leqslant \xi(y), \\ \xi(y)^{\zeta}, & l > \xi(y), \end{cases} \tag{1}$$

where A is a material-dependent constant parameter and $\xi(y)$ is a cross-over length along the x axis. For window lengths below this cross-over distance, the crack profile appears to be self-affine and the surface is characterized by the local roughness exponent $\zeta_{\rm loc}$. Above this value, which depends on the distance y from the notch tip, the roughness magnitude is driven by a different exponent, namely the global roughness exponent ζ, independent of the local one. It can be noted that this type of scaling is a generalization of self-affinity, which can be reproduced simply by imposing $\zeta = \zeta_{\rm loc}$.

The cross-over length evolves as a function of the y distance according to the following equation:

$$\xi(y) = B y^{1/z}, \tag{2}$$

where z, always greater than unity by experimental observations, is called *dynamic exponent* and seems to be, according to Morel and coworkers, a material constant. However, $\xi(y)$ is bounded, because this length cannot grow above the system size L. Thus, the self-affine correlation length $\xi(y)$ attains its maximum value $\xi_{\rm max}$ at a certain distance

$$y_{\rm sat} \sim \left(\frac{\xi_{\rm max}}{B} \right)^{z}, \tag{3}$$

where B is a material-dependent constant (Morel et al., 2000, 2002, 2003). In other words, the first growth regime of the roughness for $y < y_{\rm sat}$ is followed by a stationary regime (for $y > y_{\rm sat}$) where the roughness magnitude is a constant. Equations (1)–(3) allowed the Authors (Morel et al., 2003) to obtain a good interpretation of the R-curve behaviour showed by the toughness curves. Their approach is very simple and elegant. The classical Griffith's crack propagation criterion is introduced for a single rough crack. This is obviously a simplification, because it does not consider the existence of a crack tip process zone with diffused microcracking, which is always present in quasi-brittle materials, but links directly the morphology of the main crack to the fracture mechanism.

In Morel's work, a specimen of thickness L is considered, which contains an initial crack of length $a = a_0 + \Delta a$ (with a_0 length of the initial notch) and is subjected to uniaxial tension in mode I, see Figure 1. The fracture criterion is given by the balance of the elastic energy released at the macroscale during an infinitesimal crack propagation and the energy requested by the crack to advance, creating new free surfaces at the microscale. The following energy balance can thus be written:

$$\mathcal{G}\ \delta A_{\rm p} = 2\gamma\ \delta A_{\rm r}, \tag{4}$$

where $\mathcal{G}$ is the energy release rate at the macroscale, γ is the specific surface energy defined by Griffith, $A_{\rm p}$ the projected crack surface increment at the macroscale and $A_{\rm r}$ the real crack surface increment evaluated at the microscale. Thus, energy dissipation is concentrated on the crack surface.

Considering an infinitesimal crack increment δa, the projected surface can be evaluated as $\delta A_{\rm p} = L \delta a$, whilst the real surface can be computed through the length $\psi(y)$ of the crack profile as follows:

$$\delta A_{\mathrm{r}} = \int_{\Delta a}^{\Delta a+\delta a} \psi(y)\mathrm{d}y \approx \psi(y)\delta a. \tag{5}$$

Substituting Equation (5) into Equation (4), the energy release rate at the macroscale is obtained as

$$\mathcal{G} = 2\gamma \frac{\psi(y)}{L}, \tag{6}$$

where the ratio $\psi(y)/L$ can be considered as a roughness ratio. According to the first of Equation (1) and on the basis of a simple geometric argument, the crack length is expressed by the following relation:

$$\psi(y) = L\sqrt{1 + \left(\frac{A\xi(y)^{\zeta-\zeta_{\mathrm{loc}}}}{l_0^{1-\zeta_{\mathrm{loc}}}}\right)^2}. \tag{7}$$

Substituting this expression into Equation (6), it is possible to obtain the toughness in terms of the roughness

$$G = 2\gamma\sqrt{1 + \left(\frac{A\xi(y)^{\zeta-\zeta_{\mathrm{loc}}}}{l_0^{1-\zeta_{\mathrm{loc}}}}\right)^2}. \tag{8}$$

Finally, the previous expression should be particularized for two roughness regimes: the first where roughness increases (for $y < y_{\mathrm{sat}}$) and the second one, in which it keeps a constant value (for $y \geqslant y_{\mathrm{sat}}$).

$$G = 2\gamma \begin{cases} \sqrt{1 + \left(\frac{AB^{\zeta-\zeta_{\mathrm{loc}}}}{l_0^{1-\zeta_{\mathrm{loc}}}}\right)^2 y^{\frac{2(\zeta-\zeta_{\mathrm{loc}})}{z}}}, & y < y_{\mathrm{sat}}, \\ \sqrt{1 + \left(\frac{A\xi_{\max}^{(\zeta-\zeta_{\mathrm{loc}})}}{l_0^{1-\zeta_{\mathrm{loc}}}}\right)^2}, & y \geqslant y_{\mathrm{sat}}. \end{cases} \tag{9}$$

We wish to remark that the exponent $(\zeta - \zeta_{\mathrm{loc}})/z$ should always be less than unity to obtain an R-curve. The most important remark is that Equation (9) imply a relation between the distance y_{sat} and the maximum correlation length $\xi_{\max}$, already stated in Equation (3). This remark will give us important arguments in the discussion about the interpretation of the size effects.

The final expression for the toughness permits to reproduce in a very simple and satisfactorily way the R-curve behaviour experimentally observed, by a simple fitting. In a more recent paper, Morel et al. (2003) obtain similar results by introducing a process zone where energy is dissipated by a set of microcracks that follow the same anomalous scaling. By doing so, they improve the R-curve data fitting and obtain scaling exponents closer to those obtained directly from the fracture surface analysis, although the approach is substantially the same.

3. Size Effect on Fracture Energy

So far, no objection could be raised; problems arise as soon as one tries to derive size effects from the previous approach. Before analyzing the assumptions made by

Morel and coworkers and the subsequent results, we recall the principal results on scaling of strength and toughness in quasi-brittle materials, pointing out the fundamental difference between initially cracked and uncracked specimens.

It is well known that in quasi-brittle materials fracture energy increases with the structural size, whilst tensile strength decreases. The form of the scaling law, however, and particularly its concavity (or convexity) was long debated. Only recently it was stated without any doubts that two different cases should be considered, i.e. cracked or uncracked (respectively, notched or unnotched) specimens (Carpinteri and Chiaia, 2002; Karihaloo and Xiao, 2002).

A synthetic diagram is reported in Figure 2, where the scaling behaviour for both strength and toughness of notched and unnotched specimens is summarized. The diagram can be interpreted in two different ways. Figure 2a and c (on the left) refer to the notched case, Figure 2b and d (on the right) refer to unnotched specimens. On the other hand, the upper figures refer to fracture toughness, whilst the lower to tensile strength.

It should be noted that only three of these diagrams have been theoretically justified. In fact, the upper left diagram, showing a size effect for the toughness of notched specimens, has never been proposed before the approach based on the *anomalous scaling*. The other three laws are on the contrary well-known, at least

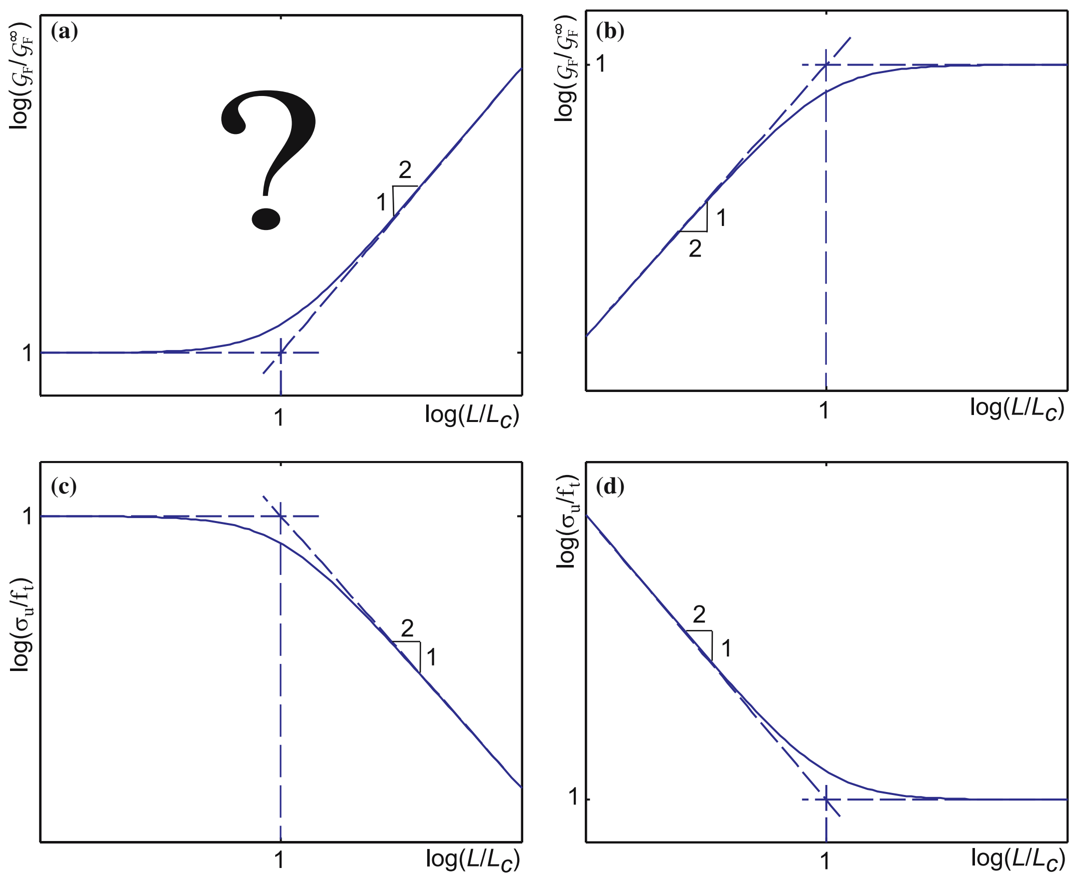

Figure 2. Scaling laws for strength and toughness of notched and unnotched specimens.

in the field of concrete structures. In fact the lower left diagram is the well-known SEL proposed by Bažant (1984), whilst the two diagrams on the right are the so-called Multi-Fractal Scaling Laws (MFSLs), proposed by Carpinteri (1994,1994a). Both of them received several confirmations, both experimental and theoretical; particularly the MFSLs have recently received a confirmation on the basis of a statistical approach, by joining statistics of extremes with stereology concepts (Carpinteri et al., 2003, 2004). Differently from what happens within the fractal framework of the MFSLs, Bažant's SEL does not provide any size effect on toughness and its derivation by means of asymptotic expansion of the energy release rate function is based on a constant toughness (Bažant, 1984).

These laws are nowadays widely accepted and even adopted by some national and international codes to account for the size-scale effects in concrete structures. Moreover, the MFSL for the tensile strength was obtained by Bažant by means of the same energy approach developed for the SEL; the two laws were incorporated in a single Universal SEL (Bažant, 1995, 1996, 1997), which encompasses any asymptotic behaviour of the two laws.

Thus, only the upper left diagram in Figure 2 did not receive a theoretical confirmation, until Morel et al. (2002a) proposed a possible explanation based on the *anomalous scaling*. In the following part of the paper, we will examine this explanation, pointing out some inconsistencies and evidencing a crucial question that has to be solved before this explanation can be accepted.

The structural size L does not appear in Equations (8) and (9). With the aim of introducing it into the toughness expression, Morel and coauthors make two assumptions, one explicit and the other implicit. These two assumptions are: (i) the maximum correlation length $\xi_{\max}$ is proportional to the structural size L (Morel et al., 2002):

$$\xi_{\max} = CL, \tag{10}$$

where the constant C was explicitly assumed to be the unity (Morel et al., 2002a); (ii) the maximum roughness, otherwise called the saturation regime, is always achieved during the fracturing process.

Both these hypotheses are questionable; regarding the former, which was corroborated by some experimental evidence (Morel et al., 1998), it should be noted that the experimental results are very weak. In fact, the ratio of the maximum to the minimum specimen size was only around six, i.e. less than one order of magnitude, which is clearly very low, especially compared with usual ratios in size effects experiments (Carpinteri and Ferro, 1994; VanVliet, 2000). This is a too narrow scale range to extrapolate scaling laws. Furthermore, in the case of spruce specimens, this ratio is reduced to less than three, with a consistent deviation of the largest specimen from the linear behaviour and the saturation regime never reached.

Furthermore, and most importantly, it is in contrast with the second assumption. In fact, substitution of Equation (10) into Equation (3) provides a new scaling law for y_{sat}:

$$y_{\mathrm{sat}} \sim \left(\frac{CL}{B}\right)^{z}. \tag{11}$$

So the distance from the notch tip at which saturation is reached should be scaled with a power law of exponent z.

As previously stated, the dynamic exponent measured by Morel et al. (1998) was in any case greater than one; thus it follows that increasing the system size, the roughness development zone before saturation increases at a faster rate than L. As a consequence, over a certain cross-over length, saturation should not be further observed. This argument is also confirmed by the previously cited experiment on spruce specimens. In fact, in the case of the largest specimen, the saturation transition was not clearly observable, due to the brief duration of the roughness map.

Clearly, supposing that the saturation regime is always reached is in contrast with Equation (11); thus it is in contrast with the successful description of the R-curve behaviour, summarized in Equation (9), which is strictly based on the formulation of the anomalous scaling, Equations (1) and (2). This contradiction can be removed only by considering a variable prefactor B in Equation (2); this point will be further analyzed in the paragraph about the size-scale effects on the tensile strength.

To point out the consequences of this assumption, let us follow to some extent the Authors' arguments. Imposing that saturation is always reached, the maximum crack growth resistance $\mathcal{G}$ is obtained substituting Equation (10) into the lower expression of Equation (9). By doing so, the following scaling law is obtained:

$$\mathcal{G} = 2\gamma\sqrt{1 + \left(\frac{A(CL)^{(\zeta-\zeta_{\mathrm{loc}})}}{l_0^{1-\zeta_{\mathrm{loc}}}}\right)^2}. \tag{12}$$

Observing its asymptotic properties, it is clear that this relation provides a scaling of the type reported in Figure 2a, presenting an horizontal asymptote at the smaller scales and a constant slope equal to $(\zeta - \zeta_{\mathrm{loc}})$ at the larger scales. The crossover length is determined from Equation (12) as:

$$L_c^{\star} = \frac{1}{C}\left(\frac{l_o^{1-\zeta_{\mathrm{loc}}}}{A}\right)^{(1/(\zeta-\zeta_{\mathrm{loc}}))}. \tag{13}$$

On the other hand, by considering Equations (11) and (12) is valid only below the transition length L_c $(>L_c^*)$, expressed by the following relation:

$$L_c = \left[\tilde{d}\left(\frac{B}{C}\right)^z\right]^{1/(z-1)}, \tag{14}$$

where $\tilde{d} = (d - a_0)/L$ is a scale-independent constant (Morel et al., 1998), see Figure 1. At the larger scales, on the contrary, the maximum crack growth resistance is obtained at the maximum crack propagation before failure. Clearly, this maximum scales with L, being exactly $(c+e)L$. Thus, substituting $y = (c+e)L$ into Equation (2) and the result into Equation (8), we obtain the following formula for the larger scales:

$$\mathcal{G} = 2\gamma\sqrt{1 + \left(\frac{A[\tilde{d}L]^{\frac{(\zeta-\zeta_{\mathrm{loc}})}{z}}}{l_0^{1-\zeta_{\mathrm{loc}}}}\right)^2}, \tag{15}$$

which is clearly different from Equation (12). In fact, the power of the scaling law at the larger scales is equal to $(\zeta - \zeta_{\mathrm{loc}})/z$ and not to $(\zeta - \zeta_{\mathrm{loc}})$. The predicted scaling thus results to be weaker, being the reported experimental values for z always greater than the unity. It should be noticed that this new formula maintains any important features of Equation (12), and particularly the property that in the case of a self-affine fracture surface, i.e. with scaling exponents $\zeta = \zeta_{\mathrm{loc}}$, no scaling appears. The resulting scaling for the crack resistance $\mathcal{G}$ is reported in Figure 3. Three stages are clearly visible: below the critical length defined by Equation (14), the scaling regime proposed by Morel – Equation (12) – holds its validity and two different slopes are observable for $L < L_c^*$ and $L > L_c^*$, respectively. Above L_c, the new scaling relation Equation (15) is valid; the asymptotic slope for the larger specimens is thus $(\zeta - \zeta_{\mathrm{loc}})/z$ and not $(\zeta - \zeta_{\mathrm{loc}})$.

It is clear that we rejected the second hypothesis and kept valid only the first one about the length ξ. Our choice is motivated by the need to preserve the validity of the model for the R-curve behaviour; in the last part of the present paper we will discuss more in detail the effects of retaining, or, respectively, rejecting, this assumption. Our choice is also motivated by the following remark on the experimental data. In the case of the spruce specimen, we collected the data for the saturation distance y_{sat} from Figure 8 in Morel et al. (2002) and performed a linear regression of its logarithm against the logarithm of the specimen size L. This is in accordance with Equation (11), which follows directly from the R-curve model. The data are collected in Table 1 and the regression line is plotted in Figure 4. The resulting value for the dynamic exponent z is greater than the unity and confirms both the experimental observations and the proposed Equation (15) for the size effect on the fracture energy. It should be noted that the obtained value is much smaller than those applied to fit the model on the experimental R-curves. In the same way, if the regression procedure is applied to the data on maritime pine specimens, the obtained estimate of the dynamical exponent z is, according to Equation (11), smaller than one, which is

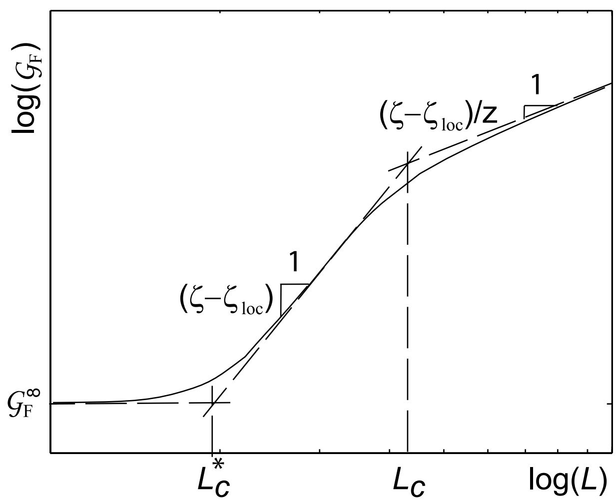

Figure 3. Size-scale effect on fracture energy, as resulting from Equation (12) (by Morel et al., 2002) and Equation (15) (present work).

Table 1. Data from Figure 8 in Morel et al. (2002).

Size L (mm)	y_{sat} (mm)
7.5	11
11.3	40
15.0	21
22.0	27
30.0	125
60.0	170

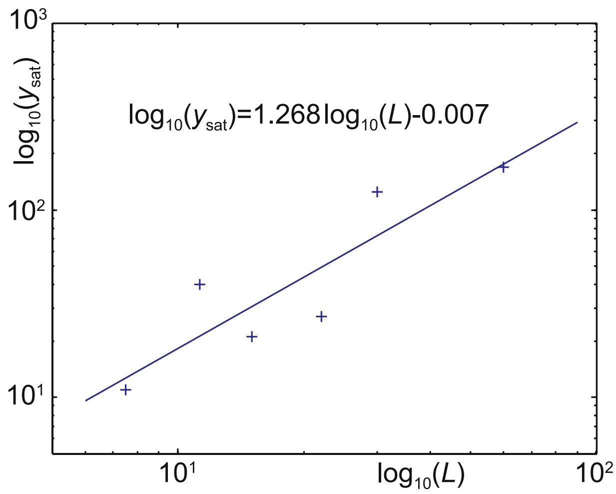

Figure 4. Regression line for saturation length y_{sat} vs. structural size L; data in Table 1, from Morel et al. (2002).

clearly meaningless! It should, however, be noted that these data are only a part of the whole data set; the discrepancy could therefore be not so great.

This discrepancy could be explained, if Equations (2) and (10) stay valid, only by introducing a variable prefactor B. The consequences of such an assumption will be considered in the following part of the paper. However, eliminating the contradiction shown by the two hypotheses, i.e. rejecting the latter and keeping valid the former expressed by Equation (10), it seems that only a slight correction on the exponent of the power law is obtained, the dynamic exponent z being not so different from one.

4. Size-Scale Effects on Tensile Strength: The Case of Notched Specimens

Given a scaling law for the fracture energy, it is possible to obtain a scaling law also for the ultimate tensile strength, within the approach developed by Bažant (1984) by means of energy release analysis and asymptotic matching. Starting from a constant fracture energy $\mathcal{G}$, he obtains the following formula for the ultimate tensile strength σ_u, according to LEFM:

$$\sigma_{\mathrm{u}} = \sqrt{\frac{E\mathcal{G}}{d g(\alpha_0, \theta)}}, \tag{16}$$

where E is the Young modulus, $\alpha_0 = a_0/L$ the initial relative crack length, $\theta = c_f/L$ a dimensionless material characteristic length and $g(\alpha, \theta)$ the dimensionless energy release rate function. From the R-curve behavior defined by Equation (9) y_{sat} can be seen as the equivalent linear elastic length of the fracture process zone (Morel et al., 2002a), i.e. $y_{\mathrm{sat}} = c_f$. Obviously, function g should be smooth so to be expanded into Taylor series with respect to the variable θ; in the case of cracked (or notched) specimens, two expressions are obtained, one valid for the larger sizes:

$$\sigma_{\mathrm{u}} = \frac{B f_t}{\sqrt{(L)}} \left(L_0^{-1} + L^{-1} + k_2 L^{-2} + k_3 L^{-3} + \cdots \right) \tag{17}$$

and the other valid for the smaller ones:

$$\sigma_{\mathrm{u}} = \sigma_P \left[1 + \left(\frac{L}{L_0} \right) + b_2 \left(\frac{L}{L_0} \right)^2 + b_3 \left(\frac{L}{L_0} \right)^3 + \cdots \right] \tag{18}$$

in which all terms, except the structural size L, are constants.

Observing the asymptotic trends given by Equations (17) and (18), it is clear that at the smaller scales the tensile strength tends to a constant maximum, whilst at the larger scales it decreases according to a power law with exponent equal to $-1/2$, i.e. according to the classical LEFM size effect. If now we consider a fracture energy which also scales with the structural size L according to Equation (12) (Morel et al., 2002a) or to Equation (15), the asymptotic trends are obtained, which are reported in Table 2.

Summarizing, at the smaller scales the tensile strength ceases to be a constant and varies according to a power law with exponent equal to $-1/2$, i.e. according to the classical LEFM size effect. The asymptote at the larger scales is again a power law, of exponent $-1/2 + (\zeta - \zeta_{\mathrm{loc}})/(2z)$. Therefore, the obtained scaling law, which is a transition between these two asymptotes, presents a curvature with upwards convexity, which disagrees with the one of Bažant's SEL. This type of scaling behaviour is suited for the case of unnotched specimens, not for the notched ones, as previously remarked (see Figure 2).

This unexpected result can be explained by this fact: in Bažant's SEL theory, the relative crack length at failure is $\alpha = \alpha_0 + \theta$, where $\theta = c_f/L$ and c_f coincides with y_{sat}. According to Equation (11), the scaling of y_{sat} implies that, for large specimen sizes, the relative crack length α at failure does not tend towards zero but towards 1 (i.e. the Fracture Process Zone (FPZ), develops over the entire ligament). In the case of small specimen sizes, the same scaling gives $\alpha \to \alpha_0$ (i.e. the FPZ tends to be only an

Table 2. Asymptotic behaviour of tensile strength; notched specimens.

	Bažant (1995)	Morel et al. (1992)	Present work
Small scales	$\sigma_{\mathrm{u}} \propto L^0$	$\sigma_{\mathrm{u}} \propto L^0$	$\sigma_{\mathrm{u}} \propto L^{-\frac{1}{2}}$
Large scales	$\sigma_{\mathrm{u}} \propto L^{-\frac{1}{2}}$	$\sigma_{\mathrm{u}} \propto L^{-\frac{1}{2}+\frac{\zeta-\zeta_{\mathrm{loc}}}{2}}$	$\sigma_{\mathrm{u}} \propto L^{-\frac{1}{2}+\frac{\zeta-\zeta_{\mathrm{loc}}}{2z}}$

infinitesimal volume near the crack tip). Both trends disagree with the hypotheses of the Bažant's SEL, for the large and also for the small specimen sizes; this fact probably pushed Morel and coworkers (2002a) to propose alternative scaling relations for y_{sat}.

For instance, at the smaller scales, the distance y_{sat}, which is a measure of the fracture process zone size, is supposed to be a constant (Morel et al., 2002a)

$$y_{\text{sat}} = \frac{n2\gamma}{\mathcal{G}_{\text{d}}} = c^*, \tag{19}$$

where $\mathcal{G}_{\text{d}}$ is a critical energy release rate per unit volume of damaged material. By keeping valid Equation (10), it is clear that the dynamic exponent z (or the prefactor B) cannot be a material constant and should vary with size. Otherwise Equation (2) is no longer valid. It has to be noted that in their papers, Morel and coworkers stated that both z (Morel et al. 2002,2002a) and B (Morel et al., 2000, 2002, 2003) are material-dependent constants.

Thus, if we assume, that both B and z are constants, a scaling law for the saturation distance y_{sat} with exponent smaller than one results in a breakdown of the fracture surface description by means of the *anomalous scaling*. It can be easily stated also graphically: with reference to Figure 5, it is evident that the scaling laws for $\xi_{\max}$ and y_{sat} are strictly related, being B and z material constants. Thus, given a scaling law for $\xi_{\max}$ in Equation (10), it is impossible considering an arbitrary scaling for y_{sat} without a breakdown of the *anomalous scaling*. Analogous considerations hold in deriving a new scaling law for y_{sat} at the larger scales.

On the other hand, the above consideration about the discrepancy between the values of z from the roughness analysis and those obtained from a regression according to Equation (11) (see Figure 4), which led us to consider a variable prefactor B, could provide the solution. A size-dependent B allows to introduce consistently different scaling laws for y_{sat}, in agreement with Bažant's hypothesis about the FPZ.

This would keep valid both the successful description of the R-curve behaviour and the size effect on the critical resistance to crack growth $\mathcal{G}$ proposed by

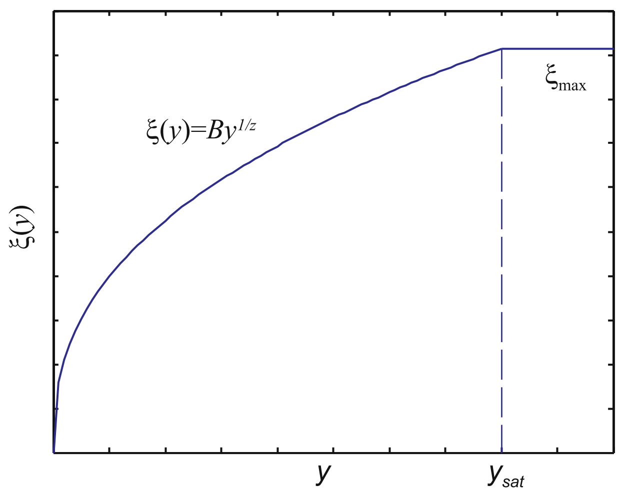

Figure 5. Correlation length according to the anomalous scaling.

(Morel et al. 2002,2002a). In our opinion, however, it is not clear if this choice is physically sound; the weak point in the above treatment could be rather the validity of Equation (10), the experimental evidence of which, as previously highlighted, should be confirmed by more extensive data.

5. Size-Scale Effects on Tensile Strength: The Case of Unnotched Specimens

An interesting task is the application of the *anomalous scaling* formulation to the case of unnotched specimens, which should provide completely different results with respect to the notched ones. In fact, any good model should capture the concavity inversion between the two cases. It is, however, very important to highlight that the anomalous scaling has never been experimentally observed in unnotched specimens but only in notched specimens; the roughness development in unnotched specimens might be different from the one described by the anomalous scaling.

Reminding that, in the case of unnotched specimens, the failure occurs at crack initiation with sudden catastrophical propagation, the calculus of the fracture energy as a function of the crack depth $a = a_0 + y$ is meaningless. On the contrary, the fracture energy of the specimen can be connected with the average roughness of the entire fracture surface. The mean value of the height–height correlation function, evaluated over the entire crack propagation length $\tilde{d}L$, is expressed by the following formula:

$$\overline{\Delta h}(l_0) = \frac{1}{\tilde{d}L}\int_0^{\tilde{d}L} \Delta h(l_0, y)\mathrm{d}y. \tag{20}$$

The corresponding mean value of the crack profile ψ results in a first approximation

$$\overline{\psi} = \frac{L}{l_0}\sqrt{l_0^2 + \overline{\Delta h}(l_0)^2} = L\sqrt{1 + \left(\frac{\overline{\Delta h}(l_0)}{l_0}\right)^2}. \tag{21}$$

Recalling Equation (4) and noting that the real and the projected areas can be rewritten, respectively, as: $A_\mathrm{r} = \overline{\psi}\tilde{d}L$ and $A_\mathrm{p} = L\tilde{d}L$, it is straightforward to obtain

$$\mathcal{G} = 2\gamma\frac{\overline{\psi}}{L} = 2\gamma\sqrt{1 + \left(\frac{\overline{\Delta h}(l_0)}{l_0}\right)^2}. \tag{22}$$

The scaling of the fracture energy is driven by the mean roughness of the entire fracture surface. To evaluate its value, two cases should be considered, depending on the condition of reached or not reached saturation. As previously discussed and emphasized, the only scaling consistent with the *anomalous scaling* is that expressed in Equation (11), otherwise a contradiction arises. By considering this type of scaling for the saturation length y_sat, two cases occur: (i) saturation reached (for $L \leqslant L_c$); (ii) saturation not reached (for $L > L_c$). Let us start from the latter case, which is simpler, to evaluate the integral of Equation (20). By substitution of Equation (1), we obtain

$$\overline{\Delta h}(l_0) = \frac{1}{\tilde{d}L}\int_0^{\tilde{d}L} A l_0^{\zeta_\mathrm{loc}} \xi(y)^{\zeta - \zeta_\mathrm{loc}}\mathrm{d}y. \tag{23}$$

In order to compute the integral, we need considering the expression of the correlation length $\xi(y)$, provided by Equation (2)

$$\overline{\Delta h}(l_0) = \frac{1}{\tilde{d}L}\int_0^{\tilde{d}L} A l_0^{\zeta_{\rm loc}} (B y^{1/z})^{\zeta-\zeta_{\rm loc}} {\rm d}y = \frac{A l_0^{\zeta_{\rm loc}} B^{\zeta-\zeta_{\rm loc}}}{\frac{\zeta-\zeta_{\rm loc}}{z}+1}\frac{1}{\tilde{d}L}(\tilde{d}L)^{1+\frac{\zeta-\zeta_{\rm loc}}{z}}. \tag{24}$$

In the former case, instead, the integral should be split into two parts, according to the *anomalous scaling* expressed by Equation (1); a first integral for the roughness development zone and a second for the constant roughness zone:

$$\overline{\Delta h}(l_0) = \frac{1}{\tilde{d}L}\int_0^{\tilde{d}L} A l_0^{\zeta_{\rm loc}} \xi(y)^{\zeta-\zeta_{\rm loc}} {\rm d}y = \frac{1}{\tilde{d}L}\left[\int_0^{y_{\rm sat}} A l_0^{\zeta_{\rm loc}} \xi(y)^{\zeta-\zeta_{\rm loc}} {\rm d}y + \int_{y_{\rm sat}}^{\tilde{d}L} A l_0^{\zeta_{\rm loc}} \xi_{\max}^{\zeta-\zeta_{\rm loc}}\, {\rm d}y\right], \tag{25}$$

where $\xi_{\max}$ is expressed in Equation (10). As we look for the asymptotic behaviour, it is evident that in the limit for $L \to 0$, the forewritten integral reduces to the following formula:

$$\overline{\Delta h}(l_0) = \frac{1}{\tilde{d}L}\int_0^{\tilde{d}L} A l_0^{\zeta_{\rm loc}} (CL)^{\zeta-\zeta_{\rm loc}} {\rm d}y, \tag{26}$$

due to the fact that, according to the scaling relation for $y_{\rm sat}$, the roughness development zone tends to zero at a faster rate than the structural size L. As a result, the first integral in Equation (25) vanishes, and the lower bound of the second integral goes to zero. The resulting mean roughness is thus expressed as

$$\overline{\Delta h}(l_0) = \frac{A l_0^{\zeta_{\rm loc}} C^{\zeta-\zeta_{\rm loc}}}{\tilde{d}L}\frac{(\tilde{d}L)^{\zeta-\zeta_{\rm loc}+1}}{\zeta-\zeta_{\rm loc}+1} = \frac{A l_0^{\zeta_{\rm loc}} (C)^{\zeta-\zeta_{\rm loc}}}{\zeta-\zeta_{\rm loc}+1}(\tilde{d}L)^{\zeta-\zeta_{\rm loc}}. \tag{27}$$

From Equations (24) and (27), it is evident that at the smaller scales the fracture energy scales with power $(\zeta-\zeta_{\rm loc})$, whilst at the larger ones it scales with a weaker slope, equal to $(\zeta-\zeta_{\rm loc})/z$. In other words, the model describes a scaling law with a downward concavity, as in the case of the MFSL by Carpinteri, depicted in Figure 2b. On the other hand, it should be noted that the horizontal asymptote of the MFSL at the larger scales is not captured by the model. Furthermore, the obtained scaling exponent for the smaller scales can be different from the value 1/2 typical of LEFM. The above results are summarized in Table 3.

The obtained scaling law for the fracture energy can be used to derive a corresponding scaling law for the ultimate tensile strength, according to Bažant's formula for the failure at crack initiation. A scaling law is obtained that correctly captures the upward concavity of Carpinteri's MFSL for the ultimate tensile strength, although it presents a positive slope for the larger sizes which is meaningless. In fact, at the smaller scales the strength decreases with increasing size according to a power law, with the exponent $[1/2-(\zeta-\zeta_{\rm loc})/2]$, whilst at the larger scales it increases with slope (in the bilogarithmic plane) equal to $(\zeta - \zeta_{loc})/z$. The asymptotic trend for the larger sizes is clearly unreliable. These results are summarized in Table 4.

It is clear that this asymptotic behaviour is a consequence of the scaling law for the maximum correlation length ξ_{max} expressed in Equation (10). This relation, as already pointed out, has been deduced from experimental results within a very narrow scale range and should be put in discussion. Admitting on the contrary that the length ξ_{max} is bounded, in order to keep bounded also the maximum roughness value, according to Equation (1), the following results are obtained: the strange asymptotic behaviour for the larger sizes in the case of unnotched specimens disappears and the horizontal asymptotes, according to the MFSL's, are obtained both for strength and toughness (see Figure 2). At the same time, however, a similar asymptotic behaviour emerges also in the case of notched specimens, and thus Equation (15) is no longer valid. In practice, the *anomalous scaling* does not allow to reproduce the size-scale effects. The hypothesis stated in Equation (10) changes drastically the obtained results and is not clear how reliable it is. We are convinced that this point should be clarified before looking for any size-scale effect on the basis of the *anomalous scaling*.

6. Conclusions

In the present paper, we investigated the connection between the description of the fracture surfaces by means of *anomalous scaling* and the so-called size-scale effects on strength and toughness of quasi-brittle materials. According to the proposed approach by Morel et al. (2002a), we link directly the fracture toughness and the tensile strength to the roughness of the single main crack, thus not considering explicitly the development of a fracture process zone during the fracturing process.

Regarding the size-scale effect on fracture energy, we showed that, in the framework of the *anomalous scaling*, the saturation regime can not be reached at the larger scales, due to the value of the dynamic exponent z, which always results greater than one. As a consequence, the power of the asymptote at the larger scales is not that proposed by Morel and coworkers, but a slightly lower value. This engenders also

Table 3. Asymptotic behaviour of fracture energy; unnotched specimens.

	Carpinteri (1994)	Present work
Small scales	$\mathcal{G} \propto L^{+\frac{1}{2}}$	$\mathcal{G} \propto L^{\zeta-\zeta_{loc}}$
Large scales	$\mathcal{G} \propto L^{0}$	$\mathcal{G} \propto L^{\frac{\zeta-\zeta_{loc}}{z}}$

Table 4. Asymptotic behaviour of tensile strength; unnotched specimens.

	Carpinteri (1994)	Present work
Small scales	$\sigma_u \propto L^{-\frac{1}{2}}$	$\sigma_u \propto L^{-\frac{1}{2}+\frac{\zeta-\zeta_{loc}}{2}}$
Large scales	$\sigma_u \propto L^{0}$	$\sigma_u \propto L^{\frac{\zeta-\zeta_{loc}}{z}}$

a slight modification of the size-scaling of tensile strength at the larger scales, the power of the asymptote being closer to the value $-1/2$ of LEFM. At the same time, this provides an unreliable concavity of the scaling law (opposite to that of Bažant's SEL), which necessarily brings to reconsider the hypotheses about the terms B and z which appear in the fundamental equations of the *anomalous scaling*.

Furthermore, we investigated the size scale effects in the case of unnotched specimens, demonstrating how the *anomalous scaling* captures only the correct concavity (respectively, convexity) of the scaling laws, whilst the asymptotic trends are not correct: the horizontal asymptotes at the larger scales are not reproduced.

Summarizing, we pointed out three unsatisfactory features in the *anomalous scaling* framework, which need to be clarified. First, the large variation in the dynamic exponent z, which is too high to fit the R-curve behaviour and too small to fit data of specimens of different sizes according to the evolution of the correlation length $\xi(y)$. Second, the anomalous scaling is at variance with Bažant's SEL hypotheses regarding the FPZ development. Both inconsistencies could be resolved by assuming a size-dependent prefactor B; within this hypothesis, the R-curve description and the size-scale effects on the fracture energy $\mathcal{G}$ proposed by Morel and coworkers are preserved.

The third feature is the strong dependency of any results on the assumption expressed by Equation (10), which is a very questionable assumption. We are convinced that this central point should be clarified, before stating the anomalous scaling provides a theoretical explanation of the upper left diagram in Figure 2, which summarizes in a unique framework the size-scale effects on strength and toughness of notched and unnotched specimens.

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