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On the asymptotic stress field in angularly nonhomogeneous materials

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Abstract. The problem of multi-material junctions composed of angularly nonhomogeneous elastic wedges in plane elasticity is addressed. For this new type of grading the governing equation for the Airy stress function is derived and, by applying the eigenfunction expansion method, a fourth-order ODE with nonconstant coefficients for the eigenequation is obtained. The solution to this ODE permits the formulation of an eigenvalue problem similar to that valid for material junctions between homogenous different materials. It is mathematically demonstrated that the angular grading influences the order of the stress-singularity. The potentials of the use of this new class of materials in joining technology are carefully investigated and some illustrative examples are deeply discussed. Comparisons with the corresponding results obtained from homogeneous materials are made.

Key words: Asymptotic analysis, fracture, functionally graded materials, interfaces, multi-material junctions

1. Introduction

Composites frequently involve situations where nonhomogeneous materials are either present naturally, or used intentionally to attain a required mechanical performance. Functionally graded materials (FGMs) are an illustrative example of two-phases synthetized materials designed in such a way that the volume fractions of the constituents vary continuously along the thickness direction to give a predetermined composition profile (Paulino, 2002). The potentials of these new material microstructures characterized by a given grading on the elastic modulus are under current investigation by the scientific community. In 1983, Erdogan stated '. . . if the crack is embedded into a nonhomogeneous medium with smoothly varying elastic properties the square root nature of the stress singularity seems to remain unchanged'. The square root singularity was mathematically proven in 1987 by Eischen under the assumption that the elastic modulus varies both radially and angularly with respect to the crack tip position at the same time:

$$E(r,\theta)=E_0\left(1+rE_1(\theta)+\frac{r^2}{2}E_2(\theta)+O(r^3)\right),\quad r\to 0, \tag{1}$$

where E_0 is a constant and $E_1(\theta), E_2(\theta)$ are smooth, bounded functions of θ. According to this expression, cracks perpendicular, parallel and arbitrarily oriented

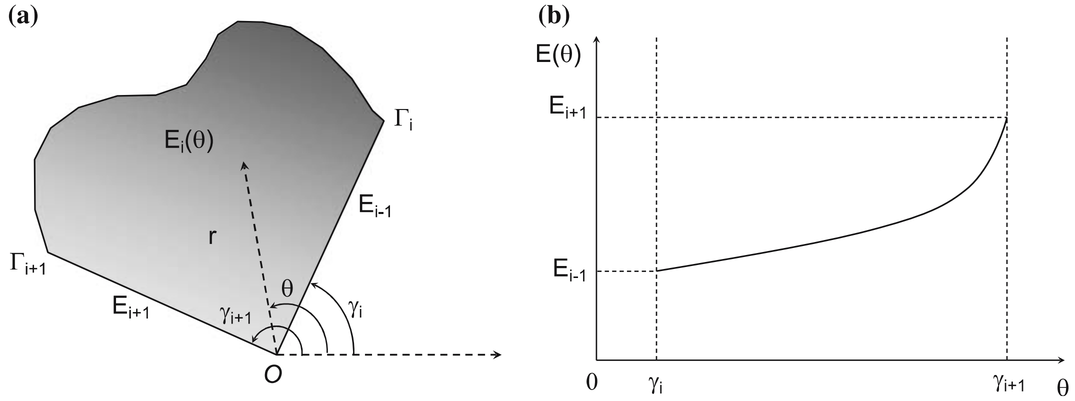

Figure 1. (a) scheme of a FGM wedge and (b) general angular variation of the elastic modulus.

with respect to the direction of the elastic gradient are situations where the nature of the square root singularity remains unchanged (Delale and Erdogan, 1983; Konda and Erdogan, 1994; Erdogan, 1995; Erdogan and Wu, 1997).

In this paper, we consider a general angular grading on the Young's modulus, i.e., $E = E(\theta)$, which was not contemplated in the mathematical analysis proposed by Eischen (1987) (see Figure 1). In fact, when the elastic modulus in Equation (1) is independent of r, it turns out to be equal to E_0, that is a constant.

The study of this type of grading is motivated by the fact that, in previous studies on bi-, tri- and multi-material junctions between different homogeneous materials (Bogy, 1968, 1971, 1974), the material parameters were assumed to be constant inside each material region and an abrupt discontinuity of the elastic properties was allowed in correspondence of zero-thickness interfaces. However, when two materials are joined together, an abrupt discontinuity of the elastic parameters, which is usually assumed for analytical purposes, does not exist in reality. On the contrary, the two materials in the proximity of the interface possess continuous, rapidly varying elastic moduli.

Hence, the governing partial differential equation for the Airy stress function is derived for this new type of grading and, for the self-consistency of the theory, it is demonstrated that the classical biharmonic condition on the Airy stress function is obtained when the elastic modulus is constant inside each material wedge. At this point, the eigenfunction expansion method (Williams, 1952) is applied and a fourth-order, linear, homogeneous ODE with nonconstant coefficients for the eigenfunction is derived. By solving this ODE, an eigenvalue problem for the characterization of the singular stress field at vertex of the material wedge can be formulated. From this result it is mathematically demonstrated that the angular grading influences the order of the stress-singularity.

According to this approach, the following problems are examined: (i) a crack inside an angularly FGM material and (ii) a tri-material junction with two homogeneous different materials joined by an intermediate FGM wedge with either perfectly bonded interfaces or with an interface crack. These configurations are important from the engineering point of view because they commonly occur in brazed tri-material systems, in the region of ply-drop in laminated composites, and at the edge close-outs in sandwich plates. The potentials of this innovative application of FGMs in joining technology is investigated, carefully emphasizing the differences in terms of the stress-singular behavior with respect to the use of the classical homogeneous counterpart.

2. Mathematical formulation

2.1. Derivation of the governing equation for the airy stress function

Let us consider an isotropic nonhomogeneous material region with an angular grading on the elastic modulus, i.e., $E = E(\theta)$, under plane strain or plane stress conditions (see Figure 1). For the sake of simplicity, the Poisson's ratio is set constant, as also assumed in Delale and Erdogan (1983), Eischen (1987), Konda and Erdogan (1994) and Erdogan (1995). This simplification does not alter any of the final conclusions of the present analysis. Neglecting body forces, stress and displacement fields can be expressed in terms of the Airy stress function $\Phi(r, \theta)$, where r and θ are polar coordinates:

$$\sigma_r = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial r^2}, \tag{2a}$$

$$\sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2}, \tag{2b}$$

$$\tau_{r\theta} = -\frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \Phi}{\partial r}, \tag{2c}$$

$$\frac{\partial u_r}{\partial r} = \frac{1}{2G}\left[\frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} - \left(1 - \frac{1}{m}\right)\nabla^2 \Phi\right], \tag{2d}$$

$$\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} = \frac{1}{G}\left[-\frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta}\right], \tag{2e}$$

where the constant m depends on the Poisson's ratio:

$$m = \begin{cases} 1 + \nu_i, & \text{for plane stress,} \\ 1/(1 - \nu_{\rm i}), & \text{for plane strain.} \end{cases} \tag{3}$$

The strain compatibility equation is given by:

$$\frac{1}{r^2}\frac{\partial^2 \epsilon_r}{\partial \theta^2} + \frac{\partial^2 \epsilon_\theta}{\partial r^2} - \frac{1}{r}\frac{\partial \epsilon_r}{\partial r} + \frac{2}{r}\frac{\partial \epsilon_\theta}{\partial r} = \frac{2}{r}\frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} + \frac{2}{r^2}\frac{\partial \gamma_{r\theta}}{\partial \theta}. \tag{4}$$

When the stress–strain relations and Equations (2a)–(2c) are substituted into Equation (4), the following governing equation for the Airy stress function in plane stress conditions is derived:

$$\nabla^4\Phi_i+\left[\frac{2}{E_i^2}\left(\frac{\mathrm{d}E_i}{\mathrm{d}\theta}\right)^2-\frac{1}{E_i}\left(\frac{\mathrm{d}^2E_i}{\mathrm{d}\theta^2}\right)\right]\left[\frac{1}{r^3}\frac{\partial\Phi_i}{\partial r}+\frac{1}{r^4}\frac{\partial^2\Phi_i}{\partial\theta^2}-\frac{\nu_i}{r^2}\frac{\partial^2\Phi_i}{\partial r^2}\right]$$
$$+\frac{1}{E_i}\frac{\mathrm{d}E_i}{\mathrm{d}\theta}\left[\frac{2\nu_i}{r^2}\frac{\partial^3\Phi_i}{\partial r^2\partial\theta}-\frac{2}{r^4}\frac{\partial^3\Phi_i}{\partial\theta^3}-\frac{2}{r^3}\frac{\partial^2\Phi_i}{\partial r\partial\theta}\right]$$
$$+\frac{1+\nu_i}{E_i}\left[\frac{\mathrm{d}E_i}{\mathrm{d}\theta}\left(-\frac{2}{r^2}\frac{\partial^3\Phi_i}{\partial r^2\partial\theta}+\frac{2}{r^3}\frac{\partial^2\Phi_i}{\partial r\partial\theta}-\frac{2}{r^4}\frac{\partial\Phi_i}{\partial\theta}\right)\right]=0, \tag{5}$$

where subscript i denotes the generic ith material region composed of a FGM.

For the self-consistency of the mathematical formulation, when the Young's modulus is constant, the classical biharmonic condition on the Airy stress function is obtained:

$$\nabla^4\Phi_i=0. \tag{6}$$

2.2. Application of the eigenfunction expansion method

At this point, the eigenfunction expansion method (Wieghardt, 1907; Williams, 1952) can be suitably applied to find out a solution to the partial differential equation (5) (see the appendix for the mathematical demonstration that the problem being considered admits a variable-separable type solution):

$$\Phi_i(r,\theta)=r^{\lambda_j+1}f_{i,j}(\theta)+r^{\lambda_j+2}g_{i,j}(\theta)+r^{\lambda_j+3}h_{i,j}(\theta)+O(r^{\lambda_j+4}), \tag{7}$$

where λ_j is the eigenvalue which at this stage is still unknown. The angular functions f, g and h are the eigenfunctions which characterize the angular variation of the stress and displacement fields. According to Eischen (1987), only the function multiplying the $r^{\lambda+1}$ term is retained in the subsequent analysis, since the other components do not contribute to the singular stress field as r approaches zero.

Upon introducing Equation (7) into Equation (5), the following fourth-order, linear, homogeneous ODE with non constant coefficients for the eigenfunction $f_{i,j}(\theta)$ is derived:

$$f_{i,j}^{\mathrm{IV}}-2\frac{E_i^{\mathrm{I}}}{E_i}f_{i,j}^{\mathrm{III}}+\left[2\left(\lambda_j^2+1\right)+2\left(\frac{E_i^{\mathrm{I}}}{E_i}\right)^2-\frac{E_i^{\mathrm{II}}}{E_i}\right]f_{i,j}^{\mathrm{II}}$$
$$+\left\{2\frac{E_i^{\mathrm{I}}}{E_i}\left[-\lambda_j^2+(\nu_i-1)\lambda_j-1\right]\right\}f_{i,j}^{\mathrm{I}}$$
$$+\left\{\left(\lambda_j^2-1\right)^2+\left(\lambda_j+1\right)\left(\lambda_j\nu_i-1\right)\left[-2\left(\frac{E_i^{\mathrm{I}}}{E_i}\right)^2+\frac{E_i^{\mathrm{II}}}{E_i}\right]\right\}f_{i,j}=0, \tag{8}$$

where primes denote derivatives with respect to θ. When $E_i(\theta)=E_i=\text{constant}$, we obtain:

$$f_{i,j}^{\mathrm{IV}}+2\left(\lambda_j^2+1\right)f_{i,j}^{\mathrm{II}}+\left(\lambda_j^2-1\right)^2f_{i,j}=0, \tag{9}$$

which corresponds to the fourth-order, linear, homogeneous ODE with constant coefficients resulting from the biharmonic condition upon the Airy stress function for isotropic homogeneous materials.

The general solution to Equation (8) has a more involved character than that of Equation (9), since the coefficients multiplying the derivatives of $f_{i,j}(\theta)$ depend on the elastic modulus and its derivatives which are, in turn, functions of the angular coordinate. It is of interest to note that if the elastic moduli have an exponential variation with respect to θ, then Equation (8) becomes a fourth-order ODE with constant coefficients. In that case, in fact, we can write (see Figure 1):

$$E_i = E_{i-1} \exp\left[\frac{1}{\gamma_{i+1}-\gamma_i} \ln\left(\frac{E_{i+1}}{E_{i-1}}\right)(\theta-\gamma_i)\right], \tag{10}$$

where E_{i-1} and E_{i+1} are, respectively, the values of the Young's moduli in correspondence of $\theta=\gamma_i$ and $\theta=\gamma_{i+1}$. Therefore, the ratios E_i^{I}/E_i and E_i^{II}/E_i are constants:

$$\frac{E_i^{\mathrm{I}}}{E_i} = \frac{1}{\gamma_{i+1}-\gamma_i} \ln\left(\frac{E_{i+1}}{E_{i-1}}\right) = k, \tag{11a}$$

$$\frac{E_i^{\mathrm{II}}}{E_i} = \left[\frac{1}{\gamma_{i+1}-\gamma_i} \ln\left(\frac{E_{i+1}}{E_{i-1}}\right)\right]^2 = k^2. \tag{11b}$$

In any case, at least theoretically, once the angular variation of the elastic modulus inside the FGM region is specified, the integral of Equation (8) can be determined within four unknown constants, as for the homogeneous case. For instance, the method of variation of parameters would be an appropriate technique in this situation for solving a problem with a general angular grading.

Finally, stress and displacement components can be expressed in terms of the eigenfunction f using Equation (2):

$$\sigma_r^i = r^{\lambda-1}\left[f_i'' + (\lambda+1) f_i\right], \tag{12a}$$

$$\sigma_\theta^i = r^{\lambda-1}\left[\lambda(\lambda+1) f_i\right], \tag{12b}$$

$$\tau_{r\theta}^i = r^{\lambda-1}\left[-\lambda f_i'\right], \tag{12c}$$

$$u_r^i = \frac{r^\lambda}{2G_i}\left\{-(\lambda+1) f_i + \frac{1}{\lambda m_i}\left[f_i'' + (\lambda+1)^2 f_i\right]\right\}, \tag{12d}$$

$$u_\theta^i = \frac{r^\lambda}{2G_i}\left\{-f_i' - \frac{1}{\lambda(\lambda-1) m_i}\left[f_i''' + (\lambda+1)^2 f_i'\right]\right\}. \tag{12e}$$

2.3. Solution for an exponentially angular variation of the Young's modulus

Let us consider the exponential form in Equation (10) as a particular type of angular variation of the Young's modulus inside the FGM wedge. As a consequence, the following fourth-order, linear, homogeneous ODE with constant coefficients has to be solved:

$$f_{i,j}^{\mathrm{IV}} - 2k f_{i,j}^{\mathrm{III}} + \left[2\left(\lambda_j^2+1\right)+k^2\right] f_{i,j}^{\mathrm{II}} - 2k\left[\lambda_j(\lambda_j+1-\nu_i)+1\right] f_{i,j}^{\mathrm{I}}$$
$$+ \left[(\lambda_j^2-1)^2 - (\lambda_j+1)(\lambda_j\nu_i-1)k^2\right] f_{i,j} = 0, \tag{13}$$

where the constant k is defined in Equation (11a). As far as the roots of the polynomial characteristic equation corresponding to Equation (13) are concerned, the following cases can be contemplated:

1. Two pairs of complex conjugate solutions:

$$x_1 \in \mathbb{C}, \quad x_2 = \overline{x_1} \in \mathbb{C}, \quad x_3 \in \mathbb{C}, \quad x_4 = \overline{x_3} \in \mathbb{C}.$$

In this case the eigenfunction assumes the following form:

$$f_{i,j}(\theta) = A_{i,j} e^{x_1\theta} + B_{i,j} e^{x_2\theta} + C_{i,j} e^{x_3\theta} + D_{i,j} e^{x_4\theta}. \tag{14}$$

An equivalent real expression can also be used instead of the complex one:

$$f_{i,j}(\theta) = A_{i,j} \mathrm{Re}\left(e^{x_1\theta}\right) + B_{i,j} \mathrm{Im}\left(e^{x_1\theta}\right) + C_{i,j} \mathrm{Re}\left(e^{x_3\theta}\right) + D_{i,j} \mathrm{Im}\left(e^{x_3\theta}\right). \tag{15}$$

This type of solution is encountered when the Young's modulus is constant and the classical biharmonic condition on the Airy stress function has to be fulfilled. In this case the roots of the characteristic equation are in fact equal to $x_1 = (\lambda+1)\mathrm{i}$, $x_2 = \overline{x_1}$, $x_3 = (\lambda-1)\mathrm{i}$ and $x_4 = \overline{x_3}$, leading to the classical expression of the eigenfunction as reported in Williams (1952):

$$f_{i,j}(\theta) = A_{i,j} \cos\left[(\lambda_j+1)\theta\right] + B_{i,j} \sin\left[(\lambda_j+1)\theta\right] + C_{i,j} \cos\left[(\lambda_j-1)\theta\right] + D_{i,j} \sin\left[(\lambda_j-1)\theta\right]. \tag{16}$$

According to this result, the following components of stress and displacement fields can be obtained:

$$\sigma_r^{i,j} = r^{\lambda_j-1}\left[(x_1^2+\lambda_j+1)e^{x_1\theta}A_{i,j} + (x_2^2+\lambda_j+1)e^{x_2\theta}B_{i,j} + (x_3^2+\lambda_j+1)e^{x_3\theta}C_{i,j} + (x_4^2+\lambda_j+1)e^{x_4\theta}D_{i,j}\right], \tag{17a}$$

$$\sigma_\theta^{i,j} = r^{\lambda_j-1}\lambda_j(\lambda_j+1)\left[e^{x_1\theta}A_{i,j} + e^{x_2\theta}B_{i,j} + e^{x_3\theta}C_{i,j} + e^{x_4\theta}D_{i,j}\right], \tag{17b}$$

$$\tau_{r\theta}^{i,j} = -r^{\lambda_j-1}\lambda_j\left[x_1e^{x_1\theta}A_{i,j} + x_2e^{x_2\theta}B_{i,j} + x_3e^{x_3\theta}C_{i,j} + x_4e^{x_4\theta}D_{i,j}\right], \tag{17c}$$

$$u_r^{i,j} = \frac{r^{\lambda_j}}{2G_i}\left\{\left[-(\lambda_j+1)+\frac{x_1^2+(\lambda_j+1)^2}{\lambda_j m_i}\right]e^{x_1\theta}A_{i,j} + \left[-(\lambda_j+1)+\frac{x_2^2+(\lambda_j+1)^2}{\lambda_j m_i}\right]e^{x_2\theta}B_{i,j} + \left[-(\lambda_j+1)+\frac{x_3^2+(\lambda_j+1)^2}{\lambda_j m_i}\right]e^{x_3\theta}C_{i,j} + \left[-(\lambda_j+1)+\frac{x_4^2+(\lambda_j+1)^2}{\lambda_j m_i}\right]e^{x_4\theta}D_{i,j}\right\} \tag{17d}$$

$$u_\theta^{i,j} = \frac{r^{\lambda_j}}{2G_i}\left\{\left[-x_1 - \frac{x_1^3+(\lambda_j+1)^2 x_1}{\lambda_j(\lambda_j-1)m_i}\right]e^{x_1\theta}A_{i,j}\right.$$
$$+\left[-x_2 - \frac{x_2^3+(\lambda_j+1)^2 x_2}{\lambda_j(\lambda_j-1)m_i}\right]e^{x_2\theta}B_{i,j}$$
$$+\left[-x_3 - \frac{x_3^3+(\lambda_j+1)^2 x_3}{\lambda_j(\lambda_j-1)m_i}\right]e^{x_3\theta}C_{i,j}$$
$$\left.+\left[-x_4 - \frac{x_4^3+(\lambda_j+1)^2 x_4}{\lambda_j(\lambda_j-1)m_i}\right]e^{x_4\theta}D_{i,j}\right\}. \quad (17e)$$

2. One pair of complex conjugate solutions and two distinct real solutions:

$$x_1 \in \mathbb{C}, \quad x_2 = \overline{x_1} \in \mathbb{C}, \quad x_3 \in \mathbb{R}, \quad x_4 \in \mathbb{R}.$$

   In this case the solution reported in Equation (14) can be adopted and stresses and displacements in polar coordinates are given by Equation (17).
3. One pair of complex conjugate solutions and two coincident real solutions:

$$x_1 \in \mathbb{C}, \quad x_2 = \overline{x_1} \in \mathbb{C}, \quad x_3 \in \mathbb{R}, \quad x_4 = x_3 \in \mathbb{R}.$$

   In this case the following modified expression of the eigenfunction has to be considered:

$$f_{i,j}(\theta) = A_{i,j}e^{x_1\theta} + B_{i,j}e^{x_2\theta} + C_{i,j}e^{x_3\theta} + D_{i,j}\theta\, e^{x_3\theta}. \quad (18)$$

   Consequently, stress and displacement components are:

$$\sigma_r^{i,j} = r^{\lambda_j-1}\left\{(x_1^2+\lambda_j+1)e^{x_1\theta}A_{i,j}\right.$$
$$+(x_2^2+\lambda_j+1)e^{x_2\theta}B_{i,j} + (x_3^2+\lambda_j+1)e^{x_3\theta}C_{i,j}$$
$$\left.+\left[(x_3(2+x_3\theta)+(\lambda_j+1)\theta\right]e^{x_3\theta}D_{i,j}\right\}, \quad (19a)$$

$$\sigma_\theta^{i,j} = r^{\lambda_j-1}\lambda_j(\lambda_j+1)\left[e^{x_1\theta}A_{i,j} + e^{x_2\theta}B_{i,j}\right.$$
$$\left.+e^{x_3\theta}C_{i,j} + \theta e^{x_3\theta}D_{i,j}\right], \quad (19b)$$

$$\tau_{r\theta}^{i,j} = -r^{\lambda_j-1}\lambda_j\left[x_1e^{x_1\theta}A_{i,j} + x_2e^{x_2\theta}B_{i,j}\right.$$
$$\left.+x_3e^{x_3\theta}C_{i,j} + (1+x_3\theta)e^{x_3\theta}D_{i,j}\right], \quad (19c)$$

$$u_r^{i,j} = \frac{r^{\lambda_j}}{2G_i}\left\{\left[-(\lambda_j+1) + \frac{x_1^2+(\lambda_j+1)^2}{\lambda_j m_i}\right]e^{x_1\theta}A_{i,j}\right.$$
$$+\left[-(\lambda_j+1) + \frac{x_2^2+(\lambda_j+1)^2}{\lambda_j m_i}\right]e^{x_2\theta}B_{i,j}$$
$$+\left[-(\lambda_j+1) + \frac{x_3^2+(\lambda_j+1)^2}{\lambda_j m_i}\right]e^{x_3\theta}C_{i,j}$$

$$+\left[-(\lambda_j+1)\theta + \frac{x_3(2+x_3\theta)+(\lambda_j+1)^2\theta}{\lambda_j m_i}\right] e^{x_3\theta} D_{i,j}\Bigg\} \quad (19d)$$

$$u_\theta^{i,j} = \frac{r^{\lambda_j}}{2G_i}\left\{\left[-x_1 - \frac{x_1^3+(\lambda_j+1)^2 x_1}{\lambda_j(\lambda_j-1)m_i}\right] e^{x_1\theta} A_{i,j} + \left[-x_2 - \frac{x_2^3+(\lambda_j+1)^2 x_2}{\lambda_j(\lambda_j-1)m_i}\right] e^{x_2\theta} B_{i,j} + \left[-x_3 - \frac{x_3^3+(\lambda_j+1)^2 x_3}{\lambda_j(\lambda_j-1)m_i}\right] e^{x_3\theta} C_{i,j} + \left[-(1+x_3\theta) - \frac{x_3^2(3+x_3\theta)+(\lambda_j+1)^2(1+x_3\theta)}{\lambda_j(\lambda_j-1)m_i}\right] e^{x_3\theta} D_{i,j}\right\}. \quad (19e)$$

3. Asymptotic analysis of the stress field

3.1. Stress-singularities due to a crack in an angularly nonhomogeneous plate

Let us consider the case of an angularly nonhomogeneous plane in which a crack is introduced in correspondence of $\theta=0$ (see Figure 1a with $i=1, \gamma_1=0$ and $\gamma_2=2\pi$). The Young's modulus $E_1(\theta)$ is assumed to vary exponentially from E_0 for $\theta=0$ to E_2 for $\theta=2\pi$, with $E_2/E_0=10$:

$$\frac{E_1}{E_0} = \exp\left[\frac{1}{2\pi}\left(\ln\frac{E_2}{E_0}\right)\theta\right] = \exp\left[\frac{1}{2\pi}(\ln 10)\,\theta\right], \quad (20)$$

The first step of the asymptotic analysis requires the computation of the roots of the polynomial characteristic equation (13). The real and imaginary parts of these roots are depicted in Figure 2 vs Re λ in the range $0<\text{Re }\lambda<1$ of interest for stress-singular components.

Contrarily to the homogeneous case, i.e., $E_2/E_0=1$, the real parts of the roots are always different from zero and the imaginary parts plotted vs Re λ are no longer straight lines. Moreover, for $0<\text{Re }\lambda \lesssim 0.91$, we have two pairs of complex conjugate roots, whereas, for $0.91 \lesssim \text{Re }\lambda<1$, we have one pair of complex conjugate solutions and two distinct real roots. The case of Re $\lambda \simeq 0.91$ is characterized by one pair of complex conjugate solutions and two coincident real roots. Therefore, except for the case Re $\lambda \simeq 0.91$ which has to be treated separately, stress and displacement components can be computed according to Equation (17).

In case of a crack or a re-entrant corner, the boundary conditions are represented by traction-free conditions along the crack faces:

$$\sigma_\theta^1(r,\gamma_1)=0, \quad (21a)$$

$$\tau_{r\theta}^1(r,\gamma_1)=0, \quad (21b)$$

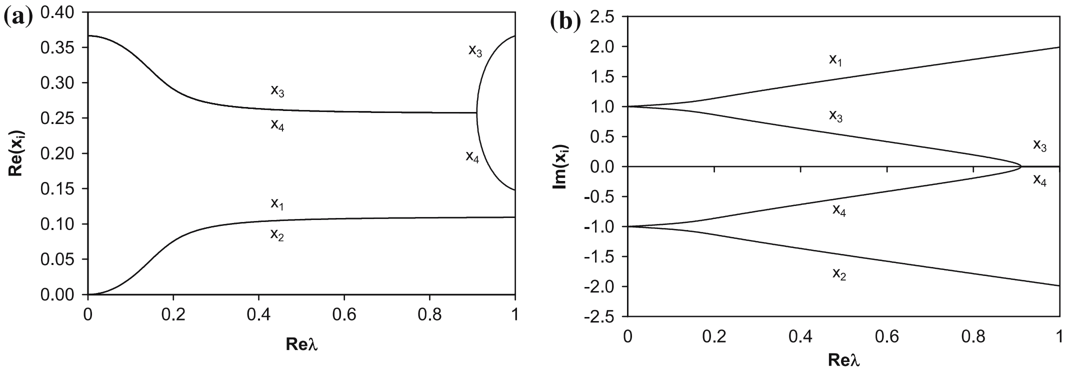

Figure 2. Roots x_i of the polynomial characteristic equation in (13) for $E_2/E_0 = 10$: (a) real and (b) imaginary parts of x_i as functions of Reλ.

$$\sigma_\theta^1(r, \gamma_2) = 0, \tag{21c}$$

$$\tau_{r\theta}^1(r, \gamma_2) = 0. \tag{21d}$$

For the present crack problem we set $\gamma_1 = 0$ and $\gamma_2 = 2\pi$. Introducing Equation (17) in the equation set (21), a matrix form $[\Lambda]\mathbf{v} = 0$ can be obtained:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ e^{2\pi x_1} & e^{2\pi x_2} & e^{2\pi x_3} & e^{2\pi x_4} \\ x_1 e^{2\pi x_1} & x_2 e^{2\pi x_2} & x_3 e^{2\pi x_3} & x_4 e^{2\pi x_4} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{22}$$

A nontrivial solution to this system exists if and only if the determinant of the coefficient matrix $[\Lambda]$ vanishes. This condition leads to an eigenequation which allows to compute the unknown eigenvalues λ. Real and imaginary parts of detΛ are depicted in Figure 3 vs Re λ for the example with $E_2/E_0 = 10$.

In the range $0 < \text{Re}\ \lambda \lesssim 0.91$ the determinant of Λ is real and always different from zero. On the other hand, in the range $0.91 \lesssim \text{Re}\ \lambda < 1$, the determinant of $[\Lambda]$ is again

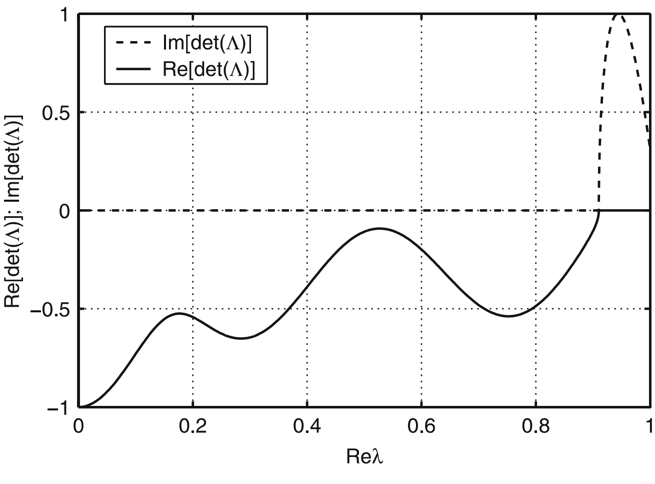

Figure 3. Real and imaginary parts of detΛ vs Re λ for a crack problem with $E_2/E_0 = 10$.

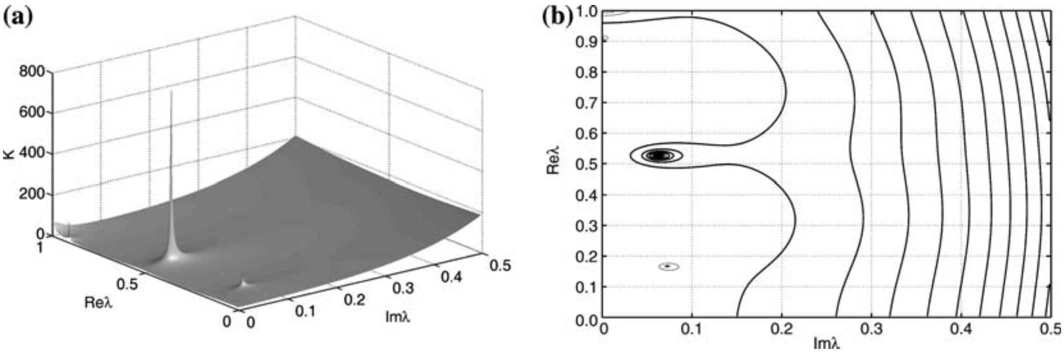

Figure 4. Plots of the condition number K vs Re λ and Imλ for a crack with $E_2/E_0 = 10$: the maximum of K is achieved for $\lambda = 0.527 + 0.063\mathrm{i}$, which represents the eigenvalue of the problem.

non singular, since its imaginary part differs from zero. The transition between these two ranges occurs exactly for Re $\lambda \simeq 0.91$. As previously discussed, it is not possible to conclude about this case using the diagram in Figure 3. For this problem, in fact, stress and displacement components have to be computed according to Equation (18) instead of Equation (17). Hence, the following modified matrix form has to be considered:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ x_1 & x_2 & x_3 & 1 \\ \mathrm{e}^{2\pi x_1} & \mathrm{e}^{2\pi x_2} & \mathrm{e}^{2\pi x_3} & 2\pi\,\mathrm{e}^{2\pi x_4} \\ x_1\mathrm{e}^{2\pi x_1} & x_2\mathrm{e}^{2\pi x_2} & x_3\mathrm{e}^{2\pi x_3} & (1+2\pi x_4)\mathrm{e}^{2\pi x_4} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{23}$$

whose determinant is different from zero and equal to $-324\mathrm{i}$. Repeating this analysis for complex values of λ, i.e., for eigenvalues in the range $0 < \mathrm{Re}\ \lambda < 1$ and $0 < \mathrm{Im}\lambda < 1$, the coefficient matrix is found to be singular for $\lambda = 0.526 + 0.063\mathrm{i}$. The corresponding condition number of $[\Lambda]$ is in fact particularly high, as expected for singular matrices (Golub and Loan, 1996; Carpinteri and Paggi, 2005a) (see Figure 4).

Considering different ratios of the Young's moduli, E_2/E_0, complex eigenvalues are found with Re $\lambda > 0.5$ (see Figure 5).

Therefore, the square root nature of the stress-singularity is modified by the effect of the elastic angular gradient. Moreover, the higher the modular ratio, the lower is the order of the stress-singularity. In fact, in the limit case consisting in $E_2/E_0 \to 0$, it is possible to analytically demonstrate that the stress-field is nonsingular. This configuration achieved when E_0 approaches infinity consists in a degenerate graded material region with $E_1 \to 0$ (see Equation (20)). Dividing Equation (13) by k^2, we recognize that the corresponding fourth-order ODE degenerates into the following second order ODE:

$$f^{\mathrm{II}} - (\lambda_j + 1)(\lambda_j \nu_i - 1) - f = 0, \tag{24}$$

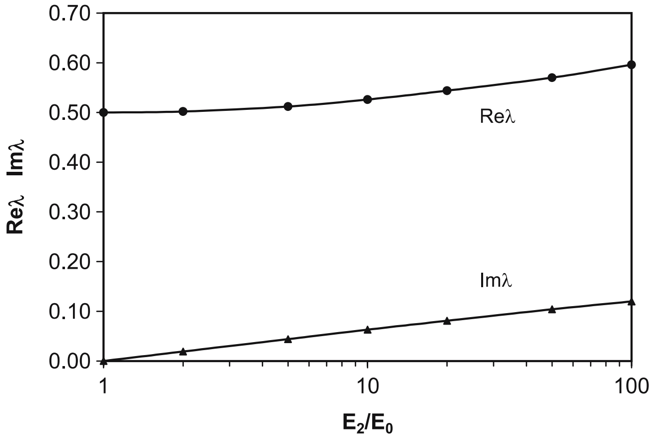

Figure 5. Real and imaginary parts of the eigenvalues leading to stress-singularities vs. E_2/E_0 for a crack problem with different angular gradients.

whose solution is:

$$f_{i,j}(\theta) = e^{x_1\theta} A_{i,j} + e^{x_2\theta} B_{i,j}, \tag{25a}$$

$$x_1 = -\sqrt{(\lambda_j + 1)(\lambda_j \nu_i - 1)}, \tag{25b}$$

$$x_2 = \sqrt{(\lambda_j + 1)(\lambda_j \nu_i - 1)}. \tag{25c}$$

For this case, stress-free boundary conditions along $\theta = 0$ and $\theta = 2\pi$ provide the same eigenequation which reads:

$$\det\Lambda = x_2 - x_1 = 2\sqrt{(\lambda_j + 1)(\lambda_j \nu_i - 1)} = 0. \tag{26}$$

This equation admits only two solutions: $\lambda_j = -1$ and $\lambda_j = 1/\nu_i$. Since we are seeking for eigenvalues such that $0 < \text{Re}\ \lambda < 1$, and the Poisson's ratio is comprised between 0 and 1/2, both solutions are unacceptable and the stress field is nonsingular.

3.2. Perfectly bonded and disbonded tri-material junctions with angularly nonhomogeneous elastic wedges

Singularity problems involving tri- and multi-material junctions have been widely investigated in the past (Theocaris, 1974; Pageau et al., 1994; Inoue and Koguchi, 1996; Carpinteri and Paggi, 2005a). In particular, tri-material junctions are commonly observed in several engineering problems, such as in brazed tri-material systems, in the region of ply-drop in laminated composites, and at the edge close-outs in sandwich plates. However, previous contributions in the literature concern junctions composed of different homogeneous materials only. In this section we consider, for the first time, tri-material junctions composed of two different homogeneous materials smoothly connected by an intermediate FGM wedge having an exponentially angular grading on the Young's modulus.

As a preliminary step, the ODE in Equation (13) has to be solved for each trial eigenvalue λ_j, with $0 < \text{Re}\ \lambda_j < 1$, in order to determine the expression of the eigenfunctions $f_{i,j}$. The form of the eigenfunction for homogeneous material wedges

Table 1. Eigenvalues computation algorithm.

LOOP	over the eigenvalues in the range $0 < \mathrm{Re}\ \lambda_j; \mathrm{Im}\lambda_j < 1$		
	LOOP	over the material regions $i = 1, \ldots, n$	
		solve the fourth-order ODE in Equation (13)	
		compute stress and displacement components according to Equation (17) or Equation (19)	
	END		
	impose BCs, obtaining the equation set $[\Lambda]\mathbf{v} = \mathbf{0}$		
	compute $\det\Lambda$		
END			
find λ_j such that $\mathrm{Re}(\det\Lambda) = \mathrm{Im}(\det\Lambda) = 0$			

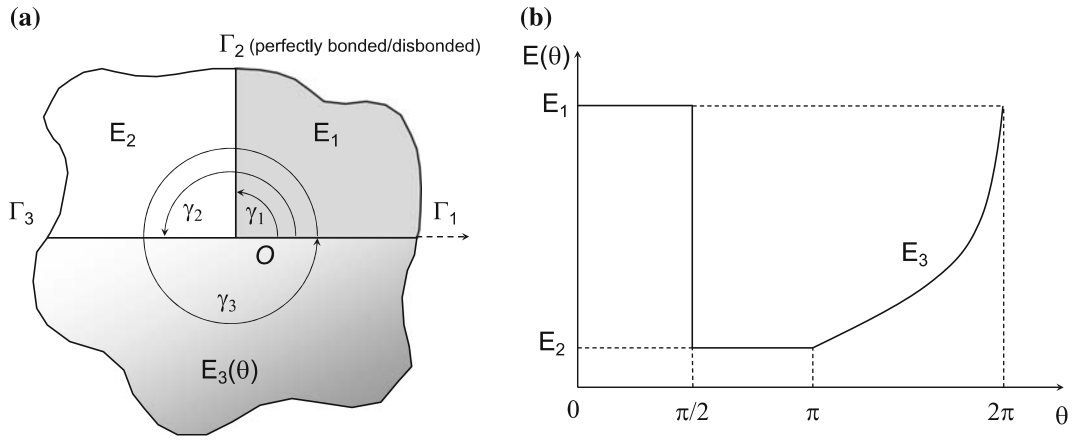

Figure 6. (a) Scheme of a tri-material junction with three material regions with perfectly bonded interfaces or with disbond along the interface between materials 1 and 2. (b) Angular variation of the elastic moduli using a FGM half plane.

reduces to the well-known expression proposed by Williams (1952) and reported in Equation (16). After that, by imposing boundary conditions along the interfaces, an equation set $[\Lambda]\mathbf{v} = 0$ can be obtained, from which we can compute the determinant of the coefficients matrix. Finally, the eigenvalues of the problem correspond to the values of λ_j which satisfy the condition of a vanishing eigenequation, i.e., $\det\Lambda = 0$. If no eigenvalues are found, the stress field is said to be nonsingular (see Table 1 for a summary of this procedure).

To fix ideas, let us consider a tri-material junction composed of two homogeneous dissimilar materials occupying two quarter plane regions joined by a FGM half plane (see Figure 6). Moreover, we assume $E_1/E_2 = 10$ and $\nu_1 = \nu_2 = \nu_3 = 0.25$ in plane stress conditions.

This geometrical configuration has been also addressed by Pageau et al. (1994), who determined the order of the stress-singularity as a function of the elastic modulus of the homogeneous half plane (material 3) for (i) perfect interface bonding conditions; and (ii) debonding along the interface between materials 1 and 2. Results of this study are reported in Figure 7 and are used as reference values for the comparison with the results of the nonhomogeneous tri-material problem.

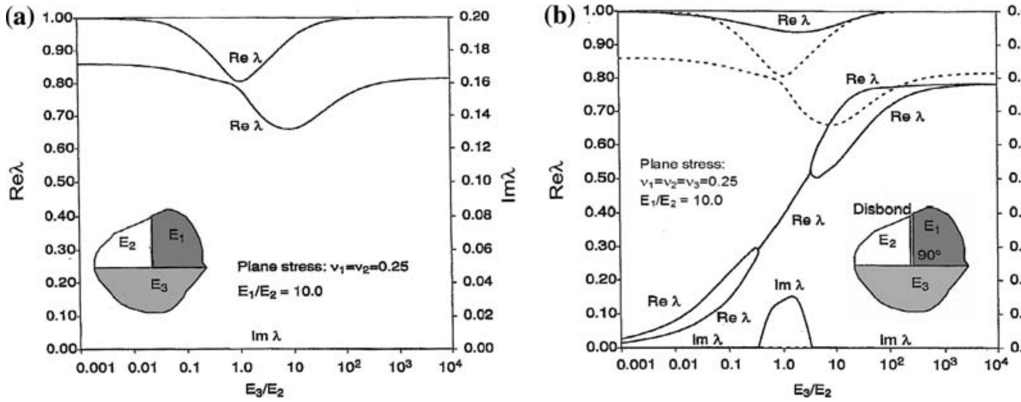

Figure 7. Eigenvalues for the homogeneous problem determined by Pageau et al. (1994) with (a) perfect bonded interfaces and (b) disbonded interface between materials 1 and 3.

For the case of perfect interface bonding, the boundary conditions are represented by stress and displacement continuity conditions along the three interfaces. For the ith interface they read as follows:

$$\sigma_\theta^i(r,\gamma_i)=\sigma_\theta^{i+1}(r,\gamma_i), \tag{27a}$$

$$\tau_{r\theta}^i(r,\gamma_i)=\tau_{r\theta}^{i+1}(r,\gamma_i), \tag{27b}$$

$$u_r^i(r,\gamma_i)=u_r^{i+1}(r,\gamma_i), \tag{27c}$$

$$u_\theta^i(r,\gamma_i)=u_\theta^{i+1}(r,\gamma_i), \tag{27d}$$

paying attention that the interface Γ_1 corresponds to $\gamma_0=0$ for region 1, and $\gamma_3=2\pi$ for region 3.

By applying the procedure summarized in Table 1, we find a real eigenvalue only equal to $\lambda_1=0.887$. This eigenvalue is higher than those corresponding to the junctions with homogeneous components for any value of the Young's modulus of material 3 (see Figure 7a). This implies that a singularity still exists but its severity is reduced.

Considering a crack along the interface between materials 1 and 2, perfect bonding boundary conditions along that interface have to be replaced by stress-free BCs, as previously discussed for a crack. Then, following the algorithm in Table 1, we find two real roots of the eigenequation equal to 0.550 and 0.625. Both eigenvalues lead to stress-singularities that are less severe than those for a crack into a homogeneous material. More importantly, the order of the stress-singularity is significantly reduced compared to the cracked problem with homogeneous dissimilar materials, where dangerous super-singularities, i.e., singularities with Re $\lambda<0.5$, occur (see Figure 7b).

4. Discussion and conclusions

The mathematical formulation for studying the asymptotic stress field at the vertex of multi-material junctions composed of angularly nonhomogeneous material wedges has been herein proposed. This study is motivated by the fact that zero-thickness interfaces are a pure mathematical model which is not confirmed by experiments. In fact, joining of two materials naturally develops transition zones where the elastic properties are rapidly varying. To model such problems, the mathematical formulation has a more involved character than that for homogeneous materials, since the bi-harmonic condition on the Airy stress function is replaced by a more complicated partial differential equation. As a consequence, the application of the eigenfunction expansion method (or the Mellin transform technique, as demonstrated in the appendix) leads to a fourth-order ODE with nonconstant coefficients. In any case, at least theoretically, once the angular variation of the elastic parameters inside the material regions is specified, the integral of that ODE can be derived. As a result, a boundary value problem similar to that for homogeneous material junctions can be formulated and solved according to the herein proposed algorithm.

Considering the special case of an exponential angular grading, two problems have been explored: (i) a crack inside an angular FGM material and (ii) a tri-material junction with a single FGM transition wedge. In the former case we have demonstrated that the order of the stress-singularity is bounded and it cannot be higher than 0.5. In the latter, it has been shown that the presence of a FGM intermediate material is favorable, since it significantly reduces the order of the stress-singularity compared to the same tri-material junction involving homogeneous different materials. This is particularly evident when an interface crack is introduced and super-singularities can be avoided.

Finally, the problem of existence or manufacturing of these angularly graded materials has to be carefully discussed. In fact, the elastic modulus of such wedge-shaped FGMs is independent of r and varies with θ, meaning that at $r=0$ it is not a single-valued function. This fact does not influence the mathematical formulation which applies for the material points for which $r \in (0, \infty)$, as also assumed in the classical multi-material junction problems.

From the physical point of view, this situation does not represent an unacceptable discontinuity for two reasons. Firstly, it has to be emphasized that such a grading on the elastic properties may naturally develop after joining of dissimilar materials. In such a case, the elastic wedges are initially homogeneous and the Young's modulus is a single-valued function inside each wedge. After joining, a multiple-valued Young's modulus at $r = 0$ is then observed, as usually occurs for multi-material junctions between homogeneous dissimilar materials. Secondly, FGMs are two-phase particulate composites synthesized in such a way that the volume fractions of the constituents vary continuously to give a predetermined composition profile. Therefore, in order to manufacture a wedge-shaped FGM, the manufacturing process should be designed to attain an angular variation of the volume fractions of the constituents. In this case the position at $r = 0$ will be occupied by a particle of one of the phases, which constitutes a material 'quantum'. Certainly, the use of such a FGM wedge in a multi-material junction does not permit to eliminate the material property discontinuity at $r = 0$. However, material discontinuities along the interfaces can be suitably removed with a favorable effect on the order of the stress-singularity. A subsequent reduction of size-scale effects due to dominant defects is also expected (Carpinteri, 1987; Carpinteri and Paggi, 2005b).

These preliminary studies clearly suggest that the use of angularly graded materials in joining technology can be very promising and further analyses should be carried out in this direction.

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Appendix

In this appendix we demonstrate, using an independent approach, that the mathematical problem considered in this paper admits a variable-separable type solution, as assumed in Equation (7) according to the application of the eigenfunction expansion method. To this aim, we apply the Mellin transform technique, deriving the solution of the problem in the transformed domain (see, e.g., Sneddon (1951) for more details about the mathematical properties of this transformation and its application to boundary value problems in polar coordinates).

We denote the Mellin transform of a function F defined and suitably regular on $(0, \infty)$ by:

$$\mathcal{M}[F, s] = \overline{F}(s) = \int_0^\infty F(r) r^{s-1}\, \mathrm{d}r, \tag{28}$$

where s is the (complex) transform parameter. Integration of Equation (28) by parts gives the Mellin transform of the derivatives of F:

$$\int_0^\infty F^n(r) r^{s-1+n}\, \mathrm{d}r = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} \overline{F}(s) \tag{29}$$

provided that $r^{s+m-1}F^{m-1}(r) \to 0$ as $r \to 0$ and $r \to \infty$ for $m = 1, 2, 3, \ldots, n$.

The formal application of the Mellin transform to Equation (5) multiplied by r^{s+3} gives the following fourth-order ODE for $\overline{\Phi}_i$:

$$\begin{aligned}
&\frac{\mathrm{d}^4\overline{\Phi}_i}{\mathrm{d}\theta^4} - 2k\frac{\mathrm{d}^3\overline{\Phi}_i}{\mathrm{d}\theta^3} + \{k^2 + [(s+2)^2 + s^2]\}\frac{\mathrm{d}^2\overline{\Phi}_i}{\mathrm{d}\theta^2} \\
&\quad + 2k[\nu_i s(s+1) + s - 2(1+\nu_i)(s+1)^2]\frac{\mathrm{d}\overline{\Phi}_i}{\mathrm{d}\theta} \\
&\quad + [s^2(s+2)^2 - sk^2 - \nu_i s(s+1)k^2]\overline{\Phi}_i = 0,
\end{aligned} \tag{30}$$

where the parameter k is defined in Equation (11a) and $\overline{\Phi}_i$ denotes the Mellin transform of the Airy stress function for the ith material wedge. Similarly, the Mellin transforms with respect to r of $r^2\sigma_\theta^i$, $r^2\tau_{r\theta}^i$, ru_r^i and ru_θ^i give:

$$\overline{r^2\sigma_\theta^i} = s(s+1)\overline{\Phi}_i,$$
$$\overline{r^2\tau_{r\theta}^i} = (s+1)\frac{\mathrm{d}\overline{\Phi}_i}{\mathrm{d}\theta},$$
$$\overline{ru_r^i} = \frac{1}{2G_i(s+1)}\left[s\left(1+s-\frac{s}{m_i}\right)\overline{\Phi}_i - \frac{1}{m_i}\frac{\mathrm{d}^2\overline{\Phi}_i}{\mathrm{d}\theta^2}\right], \tag{31}$$
$$\overline{ru_\theta^i} = \frac{1}{2G_i(s+1)(s+2)}\left[\left(s+s^2-\frac{s^2}{m_i}-2(s+1)^2\right)\frac{\mathrm{d}\overline{\Phi}_i}{\mathrm{d}\theta} - \frac{1}{m_i}\frac{\mathrm{d}^3\overline{\Phi}_i}{\mathrm{d}\theta^3}\right].$$

When the solution in the transformed domain is determined, the Mellin's inversion formula (Sneddon, 1951) can be applied to obtain the solution in the original domain. For instance, the application of the inversion formula to the Airy stress function and to the stress components gives:

$$\Phi_i(r,\theta) = \frac{1}{2\pi i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \overline{\Phi}_i r^{-s}\,\mathrm{d}s, \tag{32a}$$

$$\sigma_\theta^i(r,\theta) = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} s(s+1)\overline{\Phi}_i r^{-s-2}\,\mathrm{d}s, \tag{32b}$$

$$\sigma_r^i(r,\theta) = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \left[\frac{\mathrm{d}^2\overline{\Phi}_i}{\mathrm{d}\theta^2}\right] r^{-s-2}\mathrm{d}s, \tag{32c}$$

$$\tau_{r\theta}^i(r,\theta) = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} (s+1)\frac{\mathrm{d}\overline{\Phi}_i}{\mathrm{d}\theta} r^{-s-2}\mathrm{d}s, \tag{32d}$$

where the parameters c_i are such that $r^{c_i-1}\overline{\Phi}_i$ is absolutely integrable on $(0, \infty)$. The formal solution is now complete. From these results it is possible to recognize the following analogies between the applications of the Mellin transform technique and of the eigenfunction expansion method:

1. The ODE for $\overline{\Phi}_i$ in Equation (30) corresponds exactly to the ODE obtained for the eigenfunction f_i in Equation (13), when the parameter s is substituted with $-(\lambda+1)$.
2. Similarly, if the above mentioned change of variable is applied to Equation (31), the stresses and the displacements in the transformed domain coincide with those in Equation (12), to within the multiplicative parts that depend on r, i.e., $r^{\lambda-1}$ for the stresses and r^λ for the displacements.

As a consequence of the statements (1) and (2), it is mathematically demonstrated that the eigenvalue problem for f leads to the same results as that for $\overline{\Phi}$. The line integrals in Equation (32a-d) may be evaluated either by using the residue theorem, or by expressing them as real infinite integrals and using one of the available numerical techniques. However, to examine the behavior of the stress-singularity near the junction vertex, it is sufficient to consider the dominant terms in the infinite series obtained from the application of the residue theorem. In particular, the roots of the characteristic equation obtained by setting equal to zero the determinant of the equation system obtained from BCs provide the only singularities of the integrands in Equation (32a-d). Finally, since the integrands are functions of θ only, the inversion integral of the Airy stress function gives $\Phi \propto r^{\lambda+1} f_{ij}(\theta)$. This implies that the radial part of the solution can be separated from the angular part, thus demonstrating that the eigenfunction expansion method can be suitably applied to the problem considered in this paper.

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