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FRACTAL RULES OF MIXTURE FOR MULTI-SCALE FRAGMENTATION OF HETEROGENEOUS MATERIALS

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ABSTRACT

We propose a new approach to study the fragmentation of heterogeneous materials. It extends the fragmentation laws, deduced for homogeneous materials, to mixtures. As a practical application, we focus our attention onto artificial fragmentation by drilling of reinforced concrete in different re-bar configurations.

Keywords: fractal, fragmentation, rules of mixture, drilling, cutting

1 INTRODUCTION

Fragmentation and comminution theories have been extensively used to describe a variety of phenomena in different scientific areas [1-4]. The complexity involved in the process, due to multi-cracking interaction and propagation at different length scales, forces us to follow the process from a statistical point of view. The main parameter describing fragmentation from a global point of view can be considered the energy dissipated in the process. This statistical and energy approach has permitted to obtain universal laws for the evaluation of energy dissipation in multi-scale fragmentation due to impact or explosion [1]. The present paper extends the mentioned approach to heterogeneous materials.

Fragmentation and comminution [5] play an important role both in natural and man-made processes. Star explosion and meteor impact are examples of natural phenomena producing fragmented ejecta. Although fragmentation is of considerable importance and many experimental, numerical and theoretical studies have been carried out, relatively little progress has been made till now in developing related comprehensive theories. Fragmentation involves the interaction between fractures over a wide range of scales and a fractal fragment size distribution is expected [6].

Fractals are hierarchical, self-similar and in some cases highly irregular objects [7, 8]. As a result, no matter how complex a particular spatial pattern might be, the statistical properties of this pattern can be reproduced at different length scales. Such scale-invariant systems offer new opportunities for modelling the propagation of multiple fractures at different length scales.

Because of their complexity at any given scale, they are applicable to multiscale heterogeneous materials.

Fragmentation can occur as a result of dynamic crack propagation during compressive/tensile loading (dynamic fragmentation) or due to stress waves and their reflections during impact loading (ballistic fragmentation). These processes have been reviewed in [9-12]. Many models have been proposed to link fractals to fracture and fragmentation [13-39].

In the present paper, we propose a new approach to study the fragmentation of heterogeneous materials. It extends the fragmentation laws [1, 2], deduced for homogeneous materials and unifying the three well-known comminution theories [40-42], to mixtures. As a practical application, we have focused our attention onto artificial fragmentation by drilling of reinforced concrete. Substantially, the paper represents the conclusion of the drilling analysis proposed in [2].

2 FRACTAL FRAGMENTATION OF HETEROGENEOUS MATERIALS

The power dissipation $\dot{W}$, during the multi-scale fragmentation of a volume per unit time $\dot{V}$, for a homogeneous material can be described as [1-4]:

$$\dot{W} = \Gamma \dot{V}^{\gamma}, \qquad (1)$$

where Γ is the so-called fractal fragmentation strength (a size-independent parameter) and γ the fractal exponent, comprised between 2/3 and 1.

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For a heterogeneous material, the power dissipation for each phase i can be obtained from eq. (1):

$$\dot{W}_i = \Gamma_i \dot{V}_i^{\gamma_i} . \quad (2)$$

Since γ_i should be only slightly depending on the phase i, we can assume $\gamma_i \approx \gamma, \quad \forall i$, so that:

$$\dot{W}_i = \Gamma_i \dot{V}_i^{\gamma} . \quad (3)$$

The last hypothesis is necessary to homogenise the mixture.

3 CLASSICAL RULES OF MIXTURE

A classical fragmentation process is described by eq. (3) for $\gamma = 1$. In this case, the energy dissipation occurs in a volume. The fractal drilling strength Γ becomes the usual drilling strength $S \equiv \Gamma(\gamma = 1)$, that assumes the physical meaning of power dissipated per unit fragmented volume.

Heterogeneous structures are characterized by volume fractions, properties of the different phases, type of microstructure and load acting on their boundaries. For the derivation of their behaviour, a sequential (in series) or a simultaneous (in parallel) arrangement of microstructural components at the interface between the body inducing the fragmentation and the base materials are assumed. These are represented by the cases shown in Figures 1, 2.

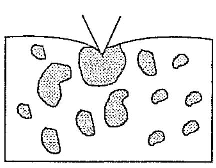

Figure 1 Two-phase heterogeneous material loaded with a punch smaller than the characteristic aggregate size.

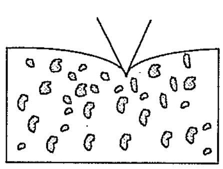

Figure 2 Two-phase heterogeneous material loaded with a punch larger than the characteristic aggregate size.

In the case of Figure 1, fragmentation occurs simply by the sequential removal of the individual components of volumetric fraction v_i and partial fragmentation strength S_i. The inverse of the macroscopic fragmentation strength S, should satisfy the following relation [43]:

$$S^{-1} = \sum_{i=1}^{N} S_i^{-1} v_i . \quad (4)$$

Eq. (4) represents the *inverse rule of mixture*.
On the other hand, assuming that the fragmentation strength may be determined by the *direct rule of mixture*, the following relation may be written [43]:

$$S = \sum_{i=1}^{N} S_i v_i . \quad (5)$$

This relation has been found for several heterogeneous materials, e.g., abrasion of fine dispersion of hard phase in soft matrix. Prerequisites are a width of the abrasive groove much larger than the particle size and spacing and a perfect bonding between the phases (Figure 2).
A third classical rule of mixture, intermediate between eqs. (4) and (5), was introduced to obtain, in some cases, a better description of the experimental data [44]:

$$S = \sum_{i=1}^{N} S_i v_i^2 . \quad (6)$$

The three rules of mixture of eqs. (4), (5) and (6) are shown in the diagram of Figure 3.

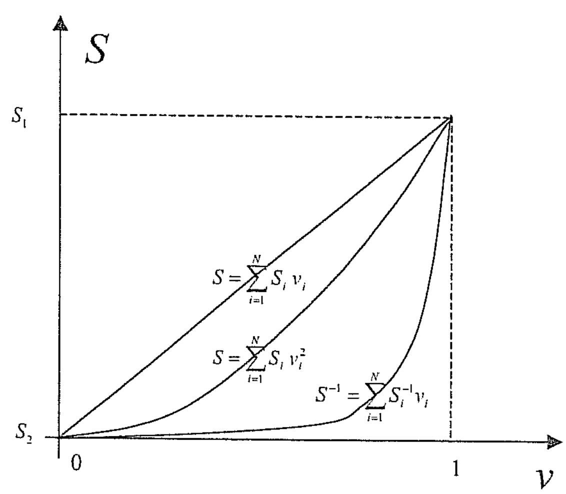

Figure 3 Rules of mixture for a two-phase material. Fragmentation strength S vs. volumetric fraction $v = v_1 = 1 - v_2$.

4 FRACTAL RULES OF MIXTURE

To determine the fractal rules of mixture for multi-scale fragmentation of heterogeneous materials, we should assume that all the phases are simultaneously fragmented. The power balance permits to obtain the power dissipation

$\dot{W}$ as the sum of the powers dissipated to crush each of the N phases:

$$\dot{W} = \sum_{i=1}^{N} \dot{W}_i = \sum_{i=1}^{N} \Gamma_i \dot{V}_i^{\gamma} . \tag{7}$$

The homogenisation of the mixture can be obtained as:

$$\dot{W} = \sum_{i=1}^{N} \Gamma_i \dot{V}_i^{\gamma} = \Gamma_{eq} \dot{V}^{\gamma} , \tag{8}$$

where Γ_{eq} is the equivalent fractal fragmentation strength:

$$\Gamma_{eq} = \sum_{i=1}^{N} \Gamma_i v_i^{\gamma} , \tag{9}$$

and the volume fractions $v_i = \frac{V_i}{V}$ satisfy the normalization rule:

$$\sum_{i=1}^{N} v_i = 1 . \tag{10}$$

Eq. (9) represents the fractal rule of mixture and, with eq. (8), describes the volume removed per unit time, $\dot{V}$, during the fragmentation of an heterogeneous material, as a function of the power $\dot{W}$ dissipated in the process:

$$\dot{V} = \left(\frac{\dot{W}}{\Gamma_{eq}} \right)^{\frac{1}{\gamma}} . \tag{11}$$

For the fragmentation of two- ($1/2 \le \gamma \le 1$) or one-dimensional ($0 \le \gamma \le 1$) heterogeneous bodies, eq. (1) becomes respectively [1]:

$$\dot{W} = \Gamma_{eq} \dot{\Omega}^{\gamma} , \tag{12}$$

$$\dot{W} = \Gamma_{eq} \dot{L}^{\gamma} , \tag{13}$$

where $\dot{\Omega}$ and $\dot{L}$ are respectively the area or length removed per unit time. The fractal rule of mixture (9) is still valid if v_i is considered respectively the area $v_i = \frac{\Omega_i}{\Omega}$ or the length $v_i = \frac{L_i}{L}$ fraction.

Note that due to the non-linearity of eq. (9), the case of an homogeneous material is recovered only for the limit case of $\gamma = 1$.

5 DRILLING COMMINUTION

We focus our attention on drilling comminution [2]. In this case, the homogenisation of the mixture is depending on the process (power consumption $\dot{W}$ or drilling velocity $\dot{\delta}$ controlled) and on the distribution of the aggregates (vertical or horizontal layers). Obviously, the first distribution in Figure 4, is the more realistic, since all the phases are distributed in vertical layers (parallel) and are simultaneously fragmented. The power consumption or drilling velocity controls coincide in this steady state process and we obtain exactly the rule of mixture of eq. (9).

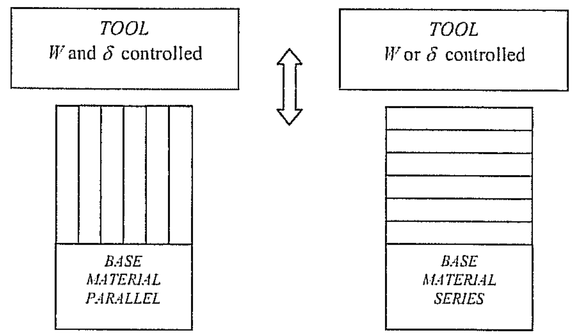

Figure 4 In parallel and in series drilling fragmentation.

On the other hand, if we assume a phase distribution in horizontal layers (in series arrangement) and a drilling velocity control ($\dot{\delta} = const$), we have:

$$\dot{W}_i = \Gamma_i \left(A\dot{\delta} \right)^{\gamma} , \tag{14}$$

A being the cross-section area of the hole and, via the energy balance:

$$W = \sum_{i=1}^{N} W_i = \sum_{i=1}^{N} \Gamma_i \left(A\dot{\delta} \right)^{\gamma} t_i , \tag{15}$$

where $t_i = \frac{h_i}{\dot{\delta}}$ is the time spent to drill the layer i of thickness h_i. The rule of mixture can be obtained comparing eq. (15) with the homogenised one:

$$W = \Gamma_{eq} \left(A\dot{\delta} \right)^{\gamma} t , \tag{16}$$

where $t = \sum_{i=1}^{N} t_i$ and being $\frac{t_i}{t} = \frac{h_i}{\sum_{i=1}^{N} h_i} = \frac{h_i}{h} = \frac{V_i}{V} = v_i$:

$$\Gamma_{eq} = \sum_{i=1}^{N} \Gamma_i v_i . \tag{17}$$

If we assume a power control ($\dot{W} = const$) and a phase distribution in horizontal layers, we have:

$$\dot{W} = \Gamma_i \left(A\dot{\delta}_i\right)^{\gamma} = \Gamma_{eq} \left(A\dot{\delta}\right)^{\gamma}, \tag{18}$$

where $\dot{\delta}$ is the mean value of the drilling velocities.
In addition, we have:

$$h = \sum_{i=1}^{N} h_i = \sum_{i=1}^{N} \dot{\delta}_i t_i \,. \tag{19}$$

Noting that:

$$h = \dot{\delta}\, t\,, \tag{20}$$

and being

$$t_i = \frac{h_i}{\dot{\delta}_i} = h_i \Gamma_i^{1/\gamma} \frac{A}{\dot{W}^{1/\gamma}}, \tag{21a}$$

$$t = \frac{h}{\dot{\delta}} = h \Gamma_{eq}^{1/\gamma} \frac{A}{\dot{W}^{1/\gamma}}, \tag{21b}$$

the rule of mixture can be obtained as:

$$\Gamma_{eq}^{1/\gamma} = \sum_{i=1}^{N} \Gamma_i^{1/\gamma} v_i \,. \tag{22}$$

To verify the obtained fractal rules of mixture, we can consider the classical (nonfractal) approach as their limit case. In this hypothesis, we have $\gamma = 1$ and $\Gamma \equiv S$, so that eq. (11) and the rules of mixture of eqs. (9), (17) and (22) become:

$$V = \frac{W}{S_{eq}}, \tag{23}$$

$$S_{eq} = \sum_{i=1}^{N} S_i v_i \,. \tag{24}$$

It is worth noting that eq. (23) represents the basic assumption to study the drilling process [45], and eq. (24) represents a direct rule of mixture [43].
Recently, Carpinteri and Pugno [2] have shown that a fractal approach for drilling comminution is more predictive than the traditional one. A multifractal extension has been also proposed by the same authors [46]. The developed fractal rules of mixture can be applied to fragmentation and comminution of heterogeneous materials. A practical application will be given in section 7.

6 GRINDABILITY

The grindability index [47] is defined as the ratio between the energy consumption in a material chosen as reference standard and the energy consumption in the tested material, when grinding the same volume to the same degree of fineness:

$$g = \frac{\dot{W}_{ref}}{\dot{W}} = \frac{\Gamma_{ref}\dot{V}^{\gamma}}{\Gamma_{eq}\dot{V}^{\gamma}} = \frac{\Gamma_{ref}}{\Gamma_{eq}}\,. \tag{25}$$

Since the grindability index g is inversely proportional to the (fractal) drilling strength, the inverse of the last one can be defined as the *grindability* of the mixture:

$$G_{eq} = \Gamma_{eq}^{-1}, \tag{26}$$

and depends on the rules of mixture of eqs. (9), (17) and (22).
Supposing $N = 2$ (binary mixtures) with $v_2 = v$ ($v_1 = 1 - v$) the rules of mixtures of eqs. (9), (17) and (22) can be plotted as reported in Figures 5. Eqs. (17) and (22) are substantially coincident and, by varying γ, always predict a worse grindability of the mixture than that of its separate phases. On the other hand, varying γ (around the unity), the more realistic rule of mixture of eq. (9) can be successfully used to model the grindability of a mixture grounds *better* ($\gamma > 1$) or *worse* ($\gamma < 1$) than that of its separate phases (Figures 5).

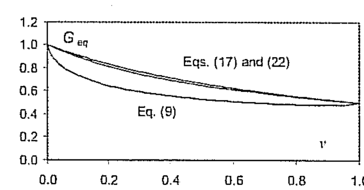

Figure 5a Grindability from the rules of mixture of eqs. (9), (17) and (22). Eqs. (17) and (22) are substantially coincident ($\Gamma_1 = 1, \Gamma_2 = 2, \gamma = 2/3$).

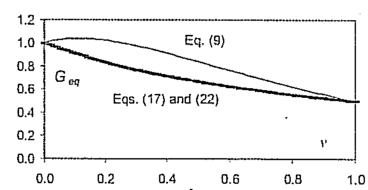

Figure 5b Grindability from the rules of mixture of eqs. (9), (17) and (22). Eqs. (17) and (22) are substantially coincident ($\Gamma_1 = 1, \Gamma_2 = 2, \gamma = 4/3$ (virtual)).

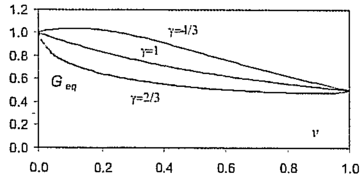

Figure 5c Grindability from the rule of mixture of eq. (9) ($\Gamma_1 = 1, \Gamma_2 = 2, \gamma = 2/3, 1, 4/3$ (virtual)).

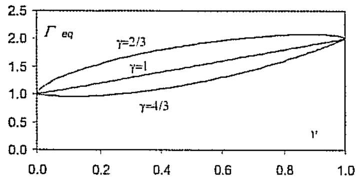

Figure 5d Fractal drilling strength (inverse of grindability) from the rule of mixture of eq. (9) ($\Gamma_1 = 1, \Gamma_2 = 2, \gamma = 2/3, 1, 4/3$ (virtual)).

Experiments show $\gamma < 1$, so that the values of $\gamma > 1$ can be considered only as virtual cases.

7 PRACTICAL EXAMPLE: DRILLING OF REINFORCED CONCRETE

We can focus our attention on the proposed fractal rule of mixture of eq. (9). For a two-phase material, it becomes:

$$\Gamma_{eq} = \Gamma_1 v_1^{2/3} + \Gamma_2 v_2^{2/3} . \tag{27}$$

2/3 being the fractal exponent for drilling comminution [2].

Some experiments are presented to validate the theoretical fractal rule of mixture of eq. (27). We have considered a special two-phase concrete, i.e., a mixture of mortar and limestone. Surprisingly, the fractal drilling strength of the two phases, mortar and limestone, is approximately the same and, therefore, of the same order of magnitude as that of the corresponding two-phase composite ($\Gamma_1 \cong \Gamma_2 \cong 15\text{MNm}^{-1}\text{s}^{-1/3}$). Eq. (27) results trivially verified. On the other hand, the fractal drilling strength of steel is about two orders of magnitude larger than that of concrete and can be easily measured ($\Gamma_s \cong 105\text{MNm}^{-1}\text{s}^{-1/3}$). We can verify eq. (27) for reinforced concrete in different configurations, like the so-called *central cut* and *banana cut*, as represented in Figure 6. The aim of this section is to predict, via the proposed rule of mixture (9), the fractal drilling strength of the corresponding mixture (reinforced concrete = concrete + re-bar).

If we consider a reinforced concrete with fractal drilling strength Γ_{rc} ($\Gamma \cong 14\text{MNm}^{-1}\text{s}^{-1/3}$ is the fractal drilling strength for plain concrete), the power being experimentally constant before, during and after cutting the re-bar (power controlled tests), we have:

$$\begin{aligned} &\dot{W} \approx \Gamma\left(A\dot{\delta}\right)^{2/3} \approx \Gamma_{rc}\left(A\dot{\delta}_{rc}\right)^{2/3} \Rightarrow \\ &\frac{\Gamma_{rc}}{\Gamma} \approx \left(\frac{\dot{\delta}}{\dot{\delta}_{rc}}\right)_{\exp}^{2/3} \approx 5^{2/3} \approx 3 \Rightarrow \quad , \\ &\Gamma_{rc} \approx 3\Gamma \approx 45\text{MNm}^{-1}\text{s}^{-1/3} \end{aligned} \tag{28}$$

the ratio of the drilling velocity in concrete (before and after cutting the re-bar) to that in reinforced concrete (during cutting the re-bar), $\dot{\delta}/\dot{\delta}_{rc}$, an experimentally measured quantity.

The experimental value of Γ_{rc} appears in agreement with the prediction of the rule of mixture (27). In fact, if v represents the volumetric fraction of steel in concrete (see Figure 6), the fractal drilling strength of the mixture can be evaluated as:

$$\Gamma_{rc} = \Gamma(1-v)^{2/3} + \Gamma_s v^{2/3} . \tag{29}$$

For the *central cut*, the volumetric fraction of steel can be estimated as (Figure 6):

$$v \approx \frac{2d}{\pi D} = 0.17 , \tag{30}$$

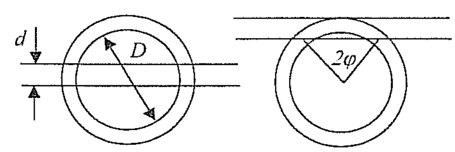

Figure 6 Central cut and banana cut configurations for reinforced concrete drilling.

where d is the diameter of the re-bar and D is the diameter of the tool.

Being $\Gamma_s \cong 105\text{MNm}^{-1}\text{s}^{-1/3}$, from the rule of mixture (29), we obtain:

$$\Gamma_{rc} = \Gamma(1-v)^{2/3} + \Gamma_s v^{2/3} \approx 44\text{MNm}^{-1}\text{s}^{-1/3} , \tag{31}$$

in agreement with the experimental result of eq. (28).

The same agreement can be found for the more critical *banana cut* configuration. In this case, the ratio between the drilling velocities, out of and in the re-bar zone, experimentally results to be twice than in the case of *central cut*:

$$\dot{W} \approx \Gamma\left(A\dot{\delta}\right)^{2/3} \approx \Gamma_{rc}\left(A\dot{\delta}_{rc}\right)^{2/3} \Rightarrow$$
$$\frac{\Gamma_{rc}}{\Gamma} \approx \left(\frac{\dot{\delta}}{\dot{\delta}_{rc}}\right)^{2/3}_{\exp} \approx 10^{2/3} \approx 4.5 \Rightarrow \tag{32}$$
$$\Gamma_{rc} \approx 4.5\Gamma \approx 63\mathrm{MNm^{-1}s^{-1/3}}$$

It means that the fractal drilling strength for the banana cut configuration experimentally results to be around 1.5 times larger than the fractal drilling strength for the *central cut* configuration.
For the banana cut configuration, if ϕ is the angular overlapping between re-bar and tool (Figure 6), we have:

$$\cos\phi \approx 1-\frac{2d}{D}, \tag{33}$$

so that:

$$v \approx \frac{\phi}{\pi} = 0.35. \tag{34}$$

From the rule of mixture of eq. (29), we have:

$$\Gamma_{rc} = \Gamma(1-v)^{2/3} + \Gamma_s v^{2/3} \approx 62\mathrm{MNm^{-1}s^{-1/3}}, \tag{35}$$

in good agreement with the experimental result of eq. (32). The experimental and theoretical results presented in this section are summarized in Table I.

Table I - Experimental and theoretical values of fractal drilling strength [$\mathrm{MNm^{-1}s^{-1/3}}$].

Concrete (Exp.)	Steel (Exp.)	RC (Exp.) Central-cut	RC (Exp.) Banana-cut	RC (Theor.) Central-cut	RC (Theor.) Banana-cut
14	105	45	44	63	62

8 CONCLUSIONS

In this paper we have proposed a new approach to study the fragmentation of mixtures. It extends the fragmentation universal laws [1] for energy dissipation in homogeneous materials to heterogeneous ones. As a practical application, we have focused our attention on artificial fragmentation by drilling of reinforced concrete (mixture of concrete and re-bar) in different configurations, like the so called *central* and *banana cut*. Theory and experiments agree satisfactorily.

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