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Numerical modelization of disordered media via fractional calculus

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Abstract

In this paper, the framework for the mechanics of solids, deformable over fractal subsets, is outlined. Anomalous mechanical quantities with fractal dimensions are introduced, i.e., the fractal stress $[\sigma^*]$, the fractal strain $[\varepsilon^*]$ and the fractal work of deformation W^*. By means of the local fractional operators, the static and kinematic equations are obtained, and the principle of virtual work for fractal media is demonstrated. Afterwards, from the definition of the fractal elastic potential ϕ^*, the linear elastic constitutive relation is derived. The direct formulation of the elastic problem is obtained in terms of the fractional Lamé operators and of the equivalence equations at the boundary. The variational form of the elastic problem is also obtained, through minimization of the total potential energy. Finally, discretization of the fractal medium is proposed, in the spirit of the Ritz–Galerkin approach, and a finite element formulation is obtained by means of devil's staircase interpolating splines.

1. Local fractional operators and the equations of statics and kinematics

Analysis of the response of a deformable body to a system of applied forces requires the definition of its constitutive law. The compatibility and equilibrium equations of continuum mechanics, which are intimately linked by the Principle of Virtual Work, need to be completed by the description of the specific material behaviour. In the presence of heterogeneous media, the classical constitutive laws (e.g. elasticity, plasticity or damage laws) need to be integrated by an appropriate modelization of the microstructure, which can considerably alter the basic formulations. This is the case, for instance, for micropolar elasticity models, higher-order gradient theories and for the large number of micromechanical models that developed in the last few years [1].

A particular class of solid materials is represented by those *deformable over a fractal subset* [2]. It is worth to notice that we are considering a generic body whose stress flux and deformation patterns have fractal characteristics, whereas the material itself does not need to have a fractal microstructure. This is the case of natural rocks and cementitious composites, which usually develop strain localization in some bands, mostly concentrated at the weak aggregate/matrix interfaces.

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The singular stress flux through fractal media can be modeled by means of lacunar fractal sets of dimension Δ_σ, with $\Delta_\sigma = 2 - d_\sigma \leqslant 2$. An original definition of the fractal stress σ^* acting upon lacunar domains was put forward by Carpinteri [3] by applying the renormalization group procedure to the nominal stress tensor $[\sigma]$. The fractal stress σ^*, whose dimensions are $[F][L]^{-(2-d_\sigma)}$, is a scale-invariant quantity. For simplicity, a uniaxial tensile field is considered in Fig. 1, where an effective fractal stress is found by means of the renormalization group.

For the local definition of σ^*, exactly as in the case of the classical Cauchy stress, the limit:

$$\lim_{\Delta A^* \to 0} (\Delta P / \Delta A^*) \tag{1}$$

is supposed to exist and to attain finite values at any singular point of the support. This is mathematically possible for lacunar sets like that in Fig. 1, which, although not compact, are dense in the surrounding of any singular point.

The kinematical conjugate of the fractal stress is the fractal strain ε^* [2,4]. The basic assumption is that displacement discontinuities can be localized on an infinite number of cross-sections, spreading throughout the body. Experimental investigations have confirmed the fractal character of deformation, for instance in metals and in highly stressed rock masses [5,6]. Considering the simplest uniaxial model, a slender bar subjected to tension, it can be argued that the horizontal projection of the cross-sections where deformation localizes is a lacunar fractal set, with dimension between zero and one. If the Cantor set ($\Delta_\varepsilon = 0.631$) is assumed as the archetype of damage distribution, we may speak of the *fractal Cantor bar* (Fig. 2(a)). The dilation strain tends to concentrate into singular stretched regions, while the rest of the body is practically undeformed. The displacement function can be represented by a *devil's staircase* graph, that is, by a singular function which is constant everywhere except at the points corresponding to a lacunar fractal set of zero Lebesgue measure (Fig. 2(b)).

Let $\Delta_\varepsilon = 1 - d_\varepsilon$ be the fractal dimension of the lacunar projection of the deformed sections. Since $\Delta_\varepsilon = 1$, the fractional decrement d_ε is always between 0.0 (corresponding to strain smeared along the bar) and 1.0 (corresponding to the maximum localization of strain, i.e., to localized fracture surfaces). By applying the renormalization group procedure (see Fig. 2(a)), the micro-scale description of displacement provides the product of an *effective* fractal strain ε^* times the fractal measure $b_0^{(1-d_\varepsilon)}$ of the support. The fractal strain ε^* is the scale-independent parameter describing the kinematics of the fractal bar. Its physical dimension $[L]^{d_\varepsilon}$ is intermediate between that of a pure strain $[L]^0$ and that of a displacement $[L]$, and synthesizes the conceptual transition between classical continuum mechanics ($d_\varepsilon = 0$) and fracture mechanics ($d_\varepsilon = 1$). Correspondingly, the kinematical controlling parameter changes, from the nominal strain ε, to the crack opening displacement w.

During a generic loading process, the mechanical work W^* can be stored in the body as elastic

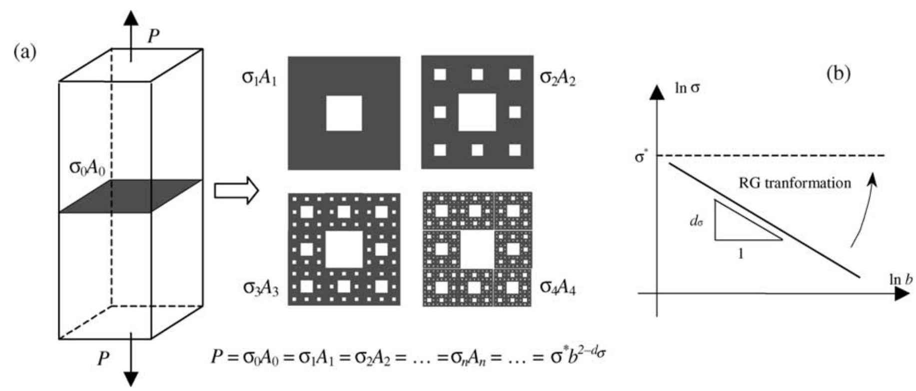

Fig. 1. Renormalization of the stress over a Sierpinski carpet (a) and scaling of the nominal stress (b).

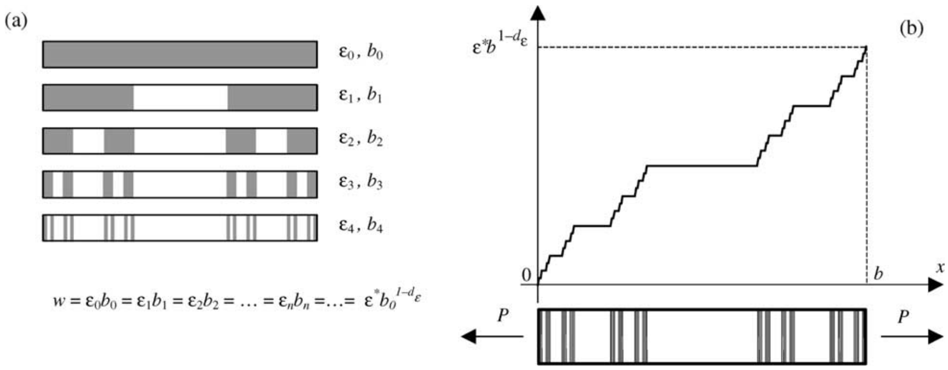

Fig. 2. Renormalization of the strain over a Cantor bar (a) and singular displacement function (b).

strain energy (conservative process) or dissipated on the infinite lacunar cross-sections where strain is localized (dissipative process). In any case, the fractal domain W^*, with dimension $\Delta_\omega = 3 - d_\omega$, where the mechanical work is produced, must be equal to the Cartesian product of the lacunar cross-section with dimension $2 - d_\sigma$, times the cantorian projection set with dimension $1 - d_\varepsilon$. Since the dimension of the Cartesian product of two fractal sets is equal to the sum of their dimensions, one obtains the fundamental relation among the exponents as:

$$d_\omega = d_\sigma + d_\varepsilon \tag{2}$$

In order to find mathematical tools suitable to work with functions and variables defined upon fractal domains, researchers started to examine the possibility of applying fractional operators, i.e. derivatives and integrals of noninteger order. Classical fractional calculus is based on nonlocal operators. On the other hand, Kolwankar and Gangal [7] have recently introduced a new operator called *local fractional integral or fractal integral*. The fractal integral is a mathematical tool suitable for the computation of fractal measures. In fact, it yields finite values if and only if the order of integration is equal to the dimension of the fractal support of the function $f(x)$. Otherwise, its value is zero or infinite, thus showing a behavior analogous to that of the Hausdorff measure of a fractal set. Kolwankar and Gangal [7] introduced also the *local fractional derivative* (LFD) of order $\alpha(0 < \alpha < 1)$ of $f(x)$:

$$[D^\alpha f(x)]_{x=y} = D^\alpha f(y) = \lim_{x\to y} \frac{\mathrm{d}^\alpha[f(x) - f(y)]}{[\mathrm{d}(x-y)]^\alpha} \tag{3}$$

where $\mathrm{d}^\alpha[f(x)]/[\mathrm{d}(x-a)]^\alpha$ denotes the (classical) Riemann–Liouville fractional derivative of order α of the function $f(x)$ choosing a as lower limit [8]. Differently from the classical fractional derivative, the LFD is a function only of the $f(x)$ values in the neighborhood of the point y where it is calculated. The classical fractional derivative of a singular function exists as long as its order is lower than the Hölder exponent characterizing the singularities. Instead, in the singular points, the LFD is generally zero or infinite. It assumes a finite value if and only if the order α of derivation is exactly equal to the Hölder exponent of the graph. For instance, in the case of the devil's staircase graph (Fig. 2(b)) the LFD of order $\alpha = \ln 2/\ln 3$ (i.e. equal to the dimension of the underlying middle-third Cantor set) is zero everywhere except in the singular points where it is finite.

By means of the LFD, the fractal differential equations of kinematics and statics have been obtained [2]. The displacement field maintains the dimension of length. The noninteger dimensions of the fractal strain are: $[L]^{d_\varepsilon}$. Therefore, it can be obtained by fractional differentiation of the displacement vector $\{\eta\}$, according to the definition of LFD outlined above. The fractional differential operator $[\partial^\alpha]$ can thus be introduced, where the order of differentiation is $\alpha = 1 - d_\varepsilon$. Thereby, the kinematic equations for the fractal medium can be written, in the usual vector notation, as:

$$\{\varepsilon^*\} = [\partial^\alpha]\{\eta\} \tag{4}$$

The above relation represents also a local definition of the fractal strain, as compared to the mean-field one provided by renormalization group (see Fig. 2). Classical nondimensional strain is obtained when $\alpha = 1 (d_\varepsilon = 0)$. Instead, when $\alpha = 0$, strain is no longer homogeneously diffused and reduces to localized displacement discontinuities. The intermediate situations are described by generic values of α.

The static equations link the fractal stress vector $\{\sigma^*\}$ to the body forces vector $\{\mathbb{F}^*\}$, which assumes noninteger dimensions according to the fractal dimension of the deformable subset Ω^*, $[F][L]^{-(3-d_\omega)}$. On the other hand, the dimensions of the fractal stress are $[F][L]^{-(2-d_\sigma)}$. Therefore, the fractal equilibrium equations can be written, in the vector notation, as:

$$[\partial^\alpha]^{\mathrm{T}}\{\sigma^*\} = -\{\mathbb{F}^*\} \tag{5}$$

where the static fractional differential operator $[\partial^\alpha]^{\mathrm{T}}$ is the transposed of the kinematic fractional differential operator $[\partial^\alpha]$. It is worth to observe that the fractional order of differentiation of the static operator in the fractal medium is $\alpha = 1 - d_\varepsilon$, the same as that of the kinematic operator. This remarkable result is due to the fundamental relation among the exponents (Eq. (2)), and represents the Duality Principle for Fractal Media [2]. Finally, equivalence at the boundary of the body requires that the stress vector coincides with the applied fractal boundary tractions $\{p^*\}$ (with physical dimensions $[F][L]^{-(2-d_\sigma)}$):

$$[\mathcal{N}]^{\mathrm{T}}\{\sigma^*\} = \{p^*\} \tag{6}$$

In the case of fractal bodies, $[\mathcal{N}]^{\mathrm{T}}$ can be defined, at any dense point of the boundary, as the cosine matrix of the outward normal to the boundary of the initiator of the fractal body.

2. Principle of Virtual Work and linear elastic constitutive law for fractal media

The Principle of Virtual Work is the fundamental identity of solid mechanics. It affirms the equality between the virtual external work (done by body forces and boundary tractions) and the virtual internal work (done by internal stresses). As is well known, the proof of the principle requires the application of the Gauss–Green Theorem. Some attempts of extending the Gauss–Green Theorem to fractal domains have been discussed in [2]. Considering two arbitrary functions $f(x,y,z)$ and $g(x,y,z)$, defined upon a fractal domain Ω^*, with the same Hölder exponent α, the general formula of *local fractional integration by parts* was obtained by the authors [2] as:

$$\{I^{(\beta-\alpha)}[gf]\}_{\Gamma^*} = \{I^\beta[gD^\alpha f]\}_{\Omega^*} + \{I^\beta[D^\alpha gf]\}_{\Omega^*} \tag{7}$$

where Γ^* is the boundary of the domain Ω^*. This result extends the Gauss–Green Theorem to 3D fractal domains. Based on Eq. (7), the Principle of Virtual Work for fractal media was demonstrated [2]. It reads:

$$\int_{\Omega^*(3-d_\omega)} \{\mathbb{F}_A^*\}^{\mathrm{T}}\{\eta_B\}\,\mathrm{d}\Omega^* + \int_{\Gamma^*(2-d_\sigma)} \{p_A^*\}^{\mathrm{T}}\{\eta_B\}\,\mathrm{d}\Gamma^* = \int_{\Omega^*(3-d_\omega)} \{\sigma_A^*\}^{\mathrm{T}}\{\varepsilon_B^*\}\,\mathrm{d}\Omega^* \tag{8}$$

Both sides of Eq. (8) possess the dimensions of work ($[F][L]$), since the operators are *fractal integrals* defined upon fractal domains. The external work may be done by fractal body forces $\{\mathbb{F}^*\}$ and/or by fractal tractions $\{p^*\}$ acting upon the boundary Γ^* of the body. The internal work of deformation is defined as: $\mathrm{d}W^* = \{\sigma^*\}^{\mathrm{T}}\{\mathrm{d}\varepsilon^*\}$, with dimensions $[F][L]^{-(2-d_\omega)}$. If the (initial) loading process is conservative (no dissipation occurs in the material), and stress is a univocal function of strain, *a fractal elastic potential* ϕ^* (function of the fractal strain $\{\varepsilon^*\}$) can be introduced as an exact differential: $\mathrm{d}\phi^* = \{\sigma^*\}^{\mathrm{T}}\{\mathrm{d}\varepsilon^*\}$. The components of the fractal stress vector $\{\sigma^*\}$ can therefore be obtained by derivation: $\sigma_i^* = \partial\phi^*/\partial\varepsilon_i^*$. Note that these are canonical first-order partial derivatives in the space of the fractal strains $\{\varepsilon^*\}$. Therefore, the same arguments of classical mathematical elasticity can be followed. Performing the Taylor expansion around the undeformed state, and neglecting higher order derivatives, the following quadratic form can be easily obtained:

$$\phi^* = \frac{1}{2}\{\varepsilon^*\}[\boldsymbol{H}^*]\{\varepsilon^*\} \tag{9}$$

where $[\boldsymbol{H}^*]$ is the Hessian matrix of the fractal elastic potential. Dimensional arguments show that the anomalous dimensions of $[\boldsymbol{H}^*]$ are: $[F][L]^{-(2+d_\varepsilon-d_\sigma)}$. Thus, $[\boldsymbol{H}^*]$ depends on both the dimensions of stress and strain. Thereby, depending on the difference $(d_\sigma - d_\varepsilon)$, the nominal elastic constants (e.g., the shear and the Young's moduli) can be subjected to positive or negative size-effects. Each term in $[\boldsymbol{H}^*]$ is obtained as the second-order partial derivative of the elastic potential by the corresponding fractal strain. From Eq. (9), the *linear elastic constitutive law for fractal media* is provided as:

$$\{\sigma^*\} = [\boldsymbol{H}^*]\{\varepsilon^*\} \tag{10}$$

3. Direct Lamé formulation of the fractal elastic problem

If the fractal constitutive law (10) is inserted into the fractal static equations (5), and using the fractal cinematic equations (4), the so-called Lamé equation is provided in the operatorial form:

$$[\mathbb{L}^*]\{\eta\} = -\{\mathbb{F}^*\}, \quad \forall P \in \Omega^* \tag{11}$$

where the fractional Lamé operator $[\mathbb{L}^*]_{3\times 3}$ is defined as: $[\mathbb{L}^*] = [\partial^\alpha]^{\mathrm{T}}[\boldsymbol{H}^*][\partial^\alpha]$. The conditions of equivalence on the boundary of the medium (Eq. (6)) where the tractions are imposed (Γ_1^*) can be expressed, as a function of the displacement vector, in the following manner:

$$[\mathbb{L}_0^*]\{\eta\} = \{p^*\}, \quad \forall P \in \Gamma_1^* \tag{12}$$

where the fractional operator $[\mathbb{L}_0^*]_{3\times 3}$ is thus defined as: $[\mathbb{L}_0^*] = [\mathcal{N}]^{\mathrm{T}}[\boldsymbol{H}^*][\partial^\alpha]$. $[\mathcal{N}^{\mathrm{T}}]$ is the cosine matrix of the outward normal to the boundary, exactly corresponding to the fractional derivatives in $[\partial^\alpha]^{\mathrm{T}}$, in the spirit of the fractal Gauss–Green Theorem. Finally, the boundary condition on the part of the boundary (Γ_2^*) with imposed displacement is the following:

$$\{\eta\} = \{\eta_0\}, \quad \forall P \in \Gamma_2^* \tag{13}$$

4. Variational formulation and discretization of the fractal elastic problem by means of devils staircase splines

The total potential energy of a fractally deformable elastic body can be defined as the difference between the strain energy stored in the body minus the external work of the body forces $\{\mathbb{F}^*\}$ and of the boundary tractions $\{p^*\}$:

$$W(\eta) = \int_{\Omega^*(3-d_\omega)} \phi^* \,\mathrm{d}\Omega^* - \left(\int_{\Omega^*(3-d_\omega)} \{\mathbb{F}^*\}^{\mathrm{T}}\{\eta\}\,\mathrm{d}\Omega^* + \int_{\Gamma^*(2-d_\sigma)} \{p^*\}^{\mathrm{T}}\{\eta\}\,\mathrm{d}\Gamma^* \right) \tag{14}$$

This functional, defined in the space of the admissible displacement and strain fields, attains its minimum value in correspondence of the (unique) solution of the fractal elastic problem [9]. This can be demonstrated by following the same arguments established for the canonical quantities, i.e., by considering an arbitrary cinematically admissible field $\{\tilde{\eta}\}$, $\{\tilde{\varepsilon}^*\}$, obtained by adding, to the fields $\{\eta\}$, $\{\varepsilon^*\}$, which are solution of the elastic problem (Eqs. (11)–(13)), a generic increment $\{\Delta\eta\}$, $\{\Delta\varepsilon^*\}$ and evaluating the functional $W(\tilde{\eta})$. If the Principle of Virtual Work (Eq. (8)) is applied to the statically admissible stress-force field (defined by Eq. (5)) working by the incremental kinematic fields $(\{\Delta\eta\}, \{\Delta\varepsilon^*\})$, using the Betti's Reciprocity Theorem one easily obtains:

$$W(\tilde{\eta}) - W(\eta) = \int_{\Omega^*(3-d_\omega)} \phi^*(\Delta\varepsilon^*)\,\mathrm{d}\Omega^* \tag{15}$$

Recalling that the fractal elastic potential is a quadratic positive form, the above relation implies that the total potential energy related to the displacement field $\{\eta\}$, solution of the elastic problem, is the minimum with respect to any other admissible field $\{\eta + \Delta\eta\}$:

$$W(\eta) = \text{minimum} \tag{16}$$

The above variational formulation is thus equivalent to the direct formulation of the elastic problem (Eqs. (11)–(13)) and represents the starting point of the Finite Element formulation. In the classical Ritz–Galerkin approach, the stationarity of the energy functional $W(\eta)$ is sought within a subspace of finite dimensions subtended by a series of linearly independent assigned functions. The problem is thus discretized, since the unknown displacement field $\{\eta\}$ is expressed as the sum of assigned independent functions $\{\eta_i\}$, $i = 1, 2, \ldots, (g \times n)$:

$$\{\eta\} = \sum_{i=1}^{(g\times n)} \beta_i \{\eta_i\} \tag{17}$$

where n represents the number of nodes where the displacement has to be determined, each with g degrees of freedom. Note, incidentally, that the physical dimension of the functions $\{\eta_i\}$ is that of length. By inserting the above relation in the definition of the total potential energy, the following functional is obtained:

$$W(\beta) = \tfrac{1}{2}\{\beta\}^{\mathrm{T}}[\boldsymbol{L}]\{\beta\} - \{\beta\}^{\mathrm{T}}[\boldsymbol{F}] \tag{18}$$

where $\{\beta\}$ is the vector of the $(g \times n)$ unknown coefficients of the linear combination (17). The $(g \times n) \times (g \times n)$ elements of the matrix $[\boldsymbol{L}]$ are given by the following fractal integrals:

$$\boldsymbol{L}_{ij} = -\int_{\Omega^*(3-d_\omega)} \{\eta_i\}^{\mathrm{T}}[\mathbb{L}^*]\{\eta_j\}\,\mathrm{d}\Omega^* + \int_{\Gamma^*(2-d_\sigma)} \{\eta_i\}^{\mathrm{T}}[\mathbb{L}_0^*]\{\eta_j\}\,\mathrm{d}\Gamma^* \tag{19}$$

Note that the matrix $[\boldsymbol{L}]$ is symmetrical due to Betti's Theorem. The (gxn) elements of the vector $[\boldsymbol{F}]$ are given by the following fractal integrals:

$$\boldsymbol{F}_{ij} = -\int_{\Omega^*(3-d_\omega)} \{\eta_i\}^{\mathrm{T}}\{\mathbb{F}^*\}\,\mathrm{d}\Omega^* + \int_{\Gamma^*(2-d_\sigma)} \{\eta_i\}^{\mathrm{T}}\{p^*\}\,\mathrm{d}\Gamma^* \tag{20}$$

Minimization of the above energy functional requires the canonical first-order derivatives of Eq. (18), with respect to each coefficient β_i, to be equal to zero. The result is the classical system of $(g \times n)$ linear algebraic equations in terms of $(g \times n)$ unknown coefficients β_i:

$$[\boldsymbol{L}]\{\beta\} = [\boldsymbol{F}] \tag{21}$$

We can choose as independent functions $\{\eta_i\}$ in the linear combination (Eq. (17)) a family of functions that, henceforth, we will call *fractal splines*. These are defined only over subsets (finite elements) of the whole fractal domain Ω^*, and their value is equal to one in correspondence of the pertinent node i and to zero in all the other nodes within their subdomain of definition (see Fig. 3). Since their linear combination gives the singular displacement field, the physical dimension of the fractal splines must be that of a length and, at the same time, they must be Cantor staircase kind functions, i.e. continuous functions everywhere constant except on a subset with null Lebesgue measure. For our purposes, the most important aspect related to the fractal splines is their ability to model the singular displacement field and to provide, by local fractional derivation, the associated strain field.

Since the fractal splines in the 1D case for finite elements with two nodes (Fig. 3) can be built connecting two devil's staircase, it is worthwhile

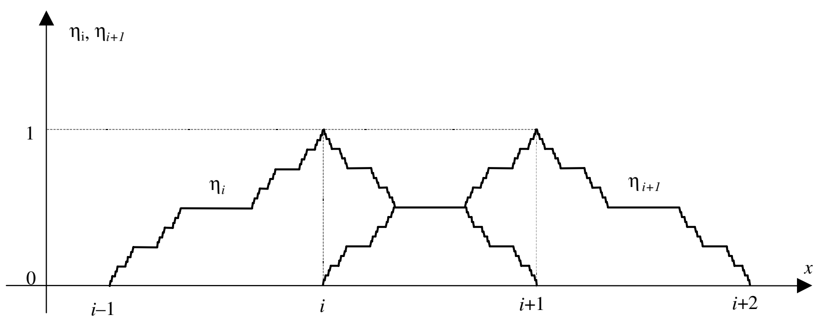

Fig. 3. Fractal splines in the 1D case for two nodes finite elements.

considering more in detail this function, archetype of the Cantor staircase kind functions, already presented in Section 1 and plotted in Fig. 2. The devil's staircase function $S(x)$ is a self-affine, monotone, nondifferentiable functions. The singularities of this function can be characterized by the Hölder exponent α lower than 1. It can be easily demonstrated that the Hölder exponent of the function $S(x)$, in all the points belonging to the Cantor set, is exactly equal to the dimension $\Delta = \ln 2/\ln 3$ of the fractal support where it grows. In other words, the devil's staircase is a function which is almost everywhere constant, except on a subset of points with dimension Δ, where it is singular of order $\alpha = \Delta$.

The classical fractional derivative (the lower extreme of integration being the origin) of the devil's staircase was studied by Schellnhuber and Seyler [10]. For these calculations, the devil's staircase can be conveniently approximated with a polygonal family of functions $S_n(x)$:

$$S_n(x) = \sum_{i=0}^{2^{n+1}-2} (-1)^i \left(\frac{3}{2}\right)^n H_{x_n(i)}(x)[x - x_n(i)] \tag{22}$$

where $H_{x_0}(x)$ is the unit step function with discontinuity in x_0 and $x_n(i)$ are the x-coordinates of the extremes of the segments of the n-th step in the iterative procedure leading to the construction of the Cantor set. It can be proved that the sequence $S_n(x)$ converges uniformly to $S(x)$ for n tending to infinity. The Riemann–Liouville fractional derivatives of the nth approximation $S_n(x)$ can be calculated to give:

$$\frac{\mathrm{d}^\alpha}{[\mathrm{d}x]^\alpha}[S_n(x)] = \sum_{i=0}^{2^{n+1}-2} (-1)^i \left(\frac{3}{2}\right)^n \frac{H_{x_n(i)}(x)}{\Gamma(2-\alpha)} \times [x - x_n(i)]^{1-\alpha} \tag{23}$$

For $\alpha < \ln 2/\ln 3$ Schellnhuber and Seyler [10] demonstrated the uniform convergence, as n tends to the infinity, of the sequence of fractional derivatives (23), thus obtaining the fractional derivative of order α of $S(x)$. For $\alpha \geqslant \ln 2/\ln 3$, the sequence (23) diverges. Thereby, the devil's staircase, although nondifferentiable in the classical sense, admits continuous fractional derivatives, provided that its order α is lower than Δ.

For our model, we are mainly interested in the local fractional derivative of the devil's staircase. Based on the classical fractional derivative (23), we can compute the local fractional derivative (LFD), for instance in the origin ([4,8]):

$$D^\alpha S(0) = \lim_{n\to\infty} \left\{ \frac{\mathrm{d}^\alpha}{[\mathrm{d}x]^\alpha}[S_n(x)] \right\}_{x=1/3^n} = \frac{1}{\Gamma(2-\alpha)} \lim_{n\to\infty} \left(\frac{3^\alpha}{2}\right)^n \tag{24}$$

Therefore, the LFD (27) in the origin is zero if $\alpha < \Delta$, does not exist if $\alpha > \Delta$, and attains a finite value if and only if $\alpha = \Delta$. The same happens in all the singular points of the devil's staircase graph, while, where the graph is flat, the LFD is zero. In other words, we have shown that the devil's staircase can be characterized by the LFD of order equal to the dimension of the fractal support where it grows.

What have been done for the function $S(x)$ can be generalized to the fractal splines (Fig. 3) and their linear combination Eq. (17). Therefore, these functions can be seen as a suitable family of functions to interpolate the singular displacement field that arises in bodies deformable on fractal subsets. Moreover, choosing as order of fractional derivation the fractal dimension of the subset where the body can get deformed, the LFD of the linear combination (17) can characterize the discontinuous strain fields of a fractally deformable medium.

Acknowledgements

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