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Computers and Structures 82 (2004) 499–508
Computers & Structures
www.elsevier.com/locate/compstruc

The elastic problem for fractal media: basic theory and finite element formulation

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Received 1 November 2002; accepted 10 October 2003

Abstract

In a previous paper [Comput. Methods Appl. Mech. Eng. 190 (2001) 6053], the framework for the mechanics of solids, deformable over fractal subsets, was outlined. Anomalous mechanical quantities with fractal dimensions were introduced, i.e., the fractal stress $[\sigma^*]$, the fractal strain $[\varepsilon^*]$ and the fractal work of deformation W^*. By means of the *local fractional operators*, the static and kinematic equations were obtained, and the Principle of Virtual Work for fractal media was demonstrated. In this paper, the constitutive equations of fractal elasticity are put forward. From the definition of the fractal elastic potential ϕ^*, the linear elastic constitutive relation is derived. The physical dimensions of the second derivatives of the elastic potential depend on the fractal dimensions of both stress and strain. Thereby, the elastic constants undergo positive or negative scaling, depending on the topological character of deformation patterns and stress flux. The direct formulation of elastic equilibrium is derived in terms of the fractional Lamé operators and of the equivalence equations at the boundary. The variational form of the elastic problem is also obtained, through minimization of the total potential energy. Finally, discretization of the fractal medium is proposed, in the spirit of the Ritz–Galerkin approach, and a finite element formulation is obtained by means of devil's staircase interpolating splines.

Keywords: Fractals; Fractional calculus; Elasticity; Finite element method; Spline approximation

1. Introduction

Analysis of the response of a deformable body to a system of applied forces requires the definition of its constitutive law. The compatibility and equilibrium equations of continuum mechanics, which are intimately linked by the Principle of Virtual Work, need to be completed by the description of the specific material behaviour. In the presence of heterogeneous media, the classical constitutive laws (e.g. elasticity, plasticity or damage laws) need to be integrated by an appropriate modelization of the microstructure, which can considerably alter the basic formulations. This is the case, for instance, for micropolar elasticity models [2], higher-order gradient theories [3] and for the large number of micromechanical models which developed in the last few years.

A particular class of solid materials is represented by those which are *deformable over a fractal subset* [1]. It is worth to notice, here, that we are considering a generic body whose stress flux and deformation patterns have fractal characteristics, whereas the material itself does not need to have a fractal microstructure. This is the case, for instance, of natural rocks and cementitious composites, which usually develop strain localization in some bands, mostly concentrated at the weak aggregate/matrix interfaces.

The mechanical consequences of fractality depend on the self-similar (or self-affine) scaling properties of the

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0045-7949/$ - see front matter
doi:10.1016/j.compstruc.2003.10.014

considered fields. For example, the well-known size-effects on the nominal mechanical properties of concrete and rocks have been successfully interpreted by the authors in this framework [4,5]. Fractality plays an important role in crack propagation, and also friction phenomena are strongly affected by fractal contact mechanisms [6].

Disorder seems to be the main characteristic of quasi-brittle material microstructure. In an attempt to model the disorder, it has been pointed out that the irregularities of the microstructure look the same at any resolution scale. This property is called self-similarity and has been detected, for instance, in concrete crack surfaces [4], as well as in rock porosity [7,8] or shear bands [9]. In general, the porosity or the disorder or the nonlinearity in the governing constitutive equations lead to the concentration of the stress on sets that are self-similar over a broad range of length scales. Therefore, a mathematical fractal set can be used to approximate the system between the lower and upper cut off length scales beyond which the system fails to be self-similar. This means that, even if difficult to handle, disorder can be described satisfactorily by means of nonclassical geometrical sets such as the fractal sets [4]. In the context of fracture of disordered media, the fractal approximations have been successfully used (see for example [10]). Numerical simulations have also been carried out to study the distribution of stress on the surface of a fractal embedded in an elastic medium [11].

The fractal patterns of deformation represent also a source of anomalous vibration modes in heterogeneous materials [12]. The propagation of stress waves in solid materials is strongly influenced by the microstructure, and in particular by its lacunarity, which is provided, at different scales, by atomic vacancies, dislocations, pores, voids and flaws. In compact structures, the presence of a well-defined characteristic length implies that the most important frequencies are involved practically at the same scale. Instead, the hierarchical structure of a fractal field is characterized by the absence of a characteristic length through several scales. This implies that the (infinite) eigenfrequencies apply to different scales, that is, they excite elements which are relatively independent of one another. Recent investigations [12] show that, in porous and heterogeneous materials, the classical wave propagation leaves place to anomalous vibration modes, highly localized, called *fractons*, which are characteristic of the highest frequencies and show an exponential decay in the space. An adequate description of these phenomena requires a rational and consistent extension of continuum mechanics and elasticity theory to fractal fields.

In a previous paper [1], the framework for the mechanics of solids, deformable over fractal subsets, has been outlined by the authors. Mechanical quantities with fractal dimensions were introduced and, by means of the *local fractional operators*, the static and kinematic equations were obtained. The Principle of Virtual Work for fractal media was finally demonstrated through an extension of the Gauss–Green Theorem to fractal sets. In this paper, the basic equations of linear elasticity are put forward for this class of fractal bodies. From the definition of *fractal elastic potential* ϕ^*, a linear elastic constitutive relation between fractal stress and fractal strain is derived. The direct formulation of the elastic problem is derived in terms of the fractional Lamé operator and of the equivalence equations at the boundary. Moreover, the variational form of the elastic problem is also obtained, through minimization of the total potential energy. Finally, discretization of the fractal medium is proposed, in the spirit of the Ritz–Galerkin approach, and a finite element formulation is obtained by means of devil's staircase interpolating splines.

2. Definition of the fractal mechanical quantities

The singular stress flux through fractal media can be modelled by means of *lacunar* fractal sets A^* of dimension Δ_σ, with $\Delta_\sigma \leqslant 2$. An original definition of the fractal stress σ^* acting upon lacunar domains was put forward by Carpinteri [4] by applying the renormalization group procedure to the nominal stress tensor $[\sigma]$. The fractal stress σ^*, whose dimensions are $[F][L]^{-(2-\Delta_\sigma)}$, is a scale-invariant quantity. For simplicity, a uniaxial tensile field is considered in Fig. 1, where an *effective* fractal stress is found by means of the renormalization group. Note that, for the local definition of σ^*, exactly as in the case of the classical Cauchy stress, the limit:

$$\lim_{\Delta A^* \to 0} (\Delta P / \Delta A^*), \tag{1}$$

is supposed to exist and to attain finite values at any singular point of the support A^*. This is mathematically possible for lacunar sets like that in Fig. 1 (and also for rarefied point sets like Cantor sets) which, although not compact, are dense in the surrounding of any singular point.

The kinematical conjugate of the fractal stress is *the fractal strain* ε^* [1,13]. The basic assumption is that displacement discontinuities can be localized on an infinite number of cross-sections, spreading throughout the body. Experimental investigations have confirmed the fractal character of deformation, for instance in metals (*slip lines* with cantorian structure [14]), and in highly stressed rock masses (*plastic shear bands* [15]).

Considering the simplest uniaxial model, a slender bar subjected to tension, it can be argued that the horizontal projection of the cross-sections where deformation localizes is a lacunar fractal set, with dimension between zero and one. If the Cantor set ($\Delta_\varepsilon \cong 0.631$) is assumed as the archetype of damage distribution, we

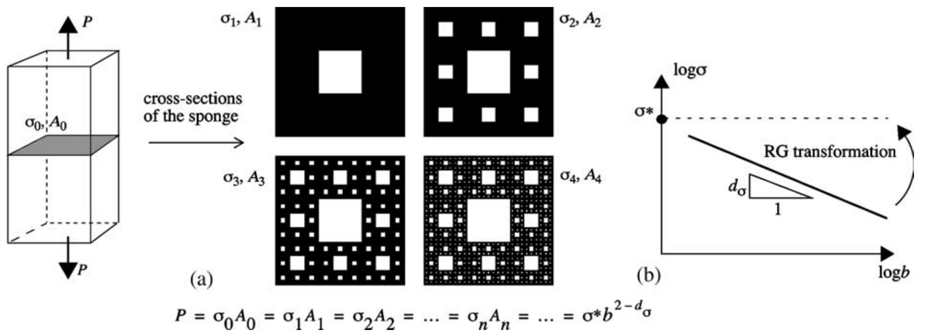

Fig. 1. Renormalization of the stress over a Sierpinski carpet (a) and scaling of the nominal stress (b).

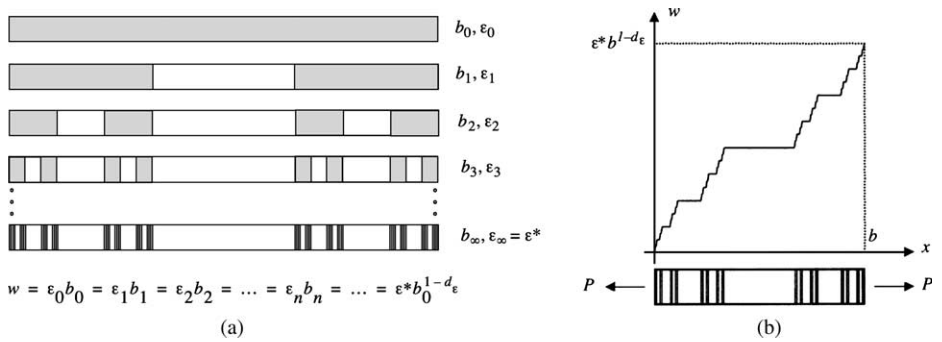

Fig. 2. Renormalization of the strain over a Cantor bar (a) and singular displacement function (b).

may speak of *the fractal Cantor bar* (Fig. 2a). The dilation strain tends to concentrate into singular stretched regions, while the rest of the body is practically undeformed. The displacement function can be represented by a *devil's staircase* graph, that is, by a singular fractal function which is constant everywhere except at the points corresponding to a lacunar fractal set of zero Lebesgue measure (Fig. 2b).

Let $\Delta_\varepsilon = 1 - d_\varepsilon$ be the fractal dimension of the lacunar projection of the deformed sections. Since $\Delta_\varepsilon \leqslant 1$, the fractional decrement d_ε is always between 0.0 (corresponding to strain smeared along the bar) and 1.0 (corresponding to the maximum localization of strain, i.e., to localized fracture surfaces). By applying the renormalization group procedure (see Fig. 2a), the microscale description of displacement provides the product of an *effective* fractal strain ε^* times the fractal measure $b_0^{(1-d_\varepsilon)}$ of the support. The fractal strain ε^* is the scale-independent parameter describing the kinematics of the fractal bar. Its physical dimensionality $[L]^{d_\varepsilon}$ is intermediate between that of a pure strain $[L]^0$ and that of a displacement [L], and synthesizes the conceptual transition between classical continuum mechanics ($d_\varepsilon = 0$) and fracture mechanics ($d_\varepsilon = 1$). Correspondingly, the kinematical controlling parameter changes, from the nominal strain ε, to the crack opening displacement w. By varying the value of d_ε (e.g. for different loading levels), the evolution of strain localization can be captured. The two limit situations are shown in Fig. 3, the devil's staircase being an intermediate situation with $d_\varepsilon \cong 0.369$. While the first case represents the classical homogeneous elastic strain field, the second diagram shows a single displacement discontinuity, e.g., the formation of a sharp fracture.

During a generic loading process, the mechanical work W^* can be stored in the body as elastic strain energy (conservative process) or dissipated on the infinite lacunar cross-sections where strain is localized (dissipative process). In any case, the fractal domain Ω^*, with dimension $\Delta_\omega = 3 - d_\omega$, where the mechanical

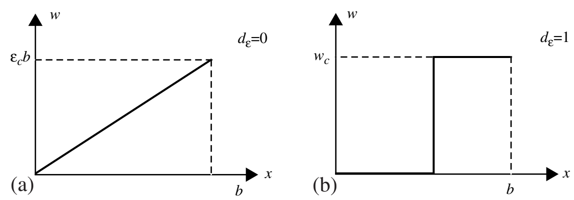

Fig. 3. Homogeneous strain (a) and extremely localized deformation (b) over the bar (critical point).

work is produced, must be equal to the Cartesian product of the lacunar cross-section with dimension $2 - d_\sigma$, times the cantorian projection set with dimension $1 - d_\varepsilon$. Since the dimension of the Cartesian product of two fractal sets is equal to the sum of their dimensions, one obtains: $(3 - d_\omega) = (2 - d_\sigma) + (1 - d_\varepsilon)$, which yields the *fundamental relation* among the exponents as:

$$d_\omega = d_\sigma + d_\varepsilon. \tag{2}$$

3. Local fractional operators and the equations of statics and kinematics

In order to find mathematical tools suitable to work with functions and variables defined upon fractal domains, researchers started to examine the possibility of applying fractional operators, i.e. derivatives and integrals of noninteger order. The application of fractional operators should provide quantities characterized by the requested noninteger physical dimensions and by peculiar scaling properties. The classical definitions of Fractional Calculus are due to Riemann and Liouville. Their definition of fractional integral can be seen as a straightforward generalization of the Cauchy integral formula. The limits of applicability of the classical approach to fractal fields have been discussed in [1].

Classical fractional calculus is based on nonlocal operators. On the other hand, Kolwankar and Gangal [16] has recently introduced a new operator called *local fractional integral* or *fractal integral*. Let $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$, $x_0 = a$, $x_N = b$, be a partition of the interval $[a, b]$, and x_i^* some suitable point of the interval $[x_i, x_{i+1}]$. Consider then a function $f(x)$ defined on a lacunar fractal set belonging to $[a, b]$. The fractal integral of order α of the function $f(x)$ over the interval $[a, b]$ is defined as:

$$I^\alpha[f(x)]_a^b = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i^*) \frac{\mathrm{d}^{-\alpha} 1_{\mathrm{d}x_i}(x)}{[\mathrm{d}(x_{i+1} - x_i)]^{-\alpha}}, \tag{3}$$

where $1_{\mathrm{d}x_i}(x)$ is the unit function defined upon $[x_i, x_{i+1}]$. The fractal integral is a mathematical tool suitable for the computation of fractal measures. In fact, it yields finite values if and only if the order of integration is equal to the dimension of the fractal support of the function $f(x)$. Otherwise, its value is zero or infinite, thus showing a behaviour analogous to that of the Hausdorff measure of a fractal set.

Kolwankar and Gangal [17,18] also introduced the *local fractional derivative* (LFD) of order α, whose definition is $(0 < \alpha < 1)$:

$$[\mathrm{D}^\alpha f(t)]_{t=x} = \mathrm{D}^\alpha f(x) = \lim_{t\to x} \frac{\mathrm{d}^\alpha[f(t) - f(x)]}{[\mathrm{d}(t - x)]^\alpha}, \tag{4}$$

where $\frac{\mathrm{d}^\alpha[f(x)]}{[\mathrm{d}(x-a)]^\alpha}$ denotes the (classical) Riemann–Liouville fractional derivative of order a of α of the function $f(x)$, choosing a as the lower integration limit [19]. Differently from the classical fractional derivative, the LFD is a function only of the $f(x)$ values in the neighborhood of the point y where it is calculated. The classical fractional derivative of a fractal function exists as long as its order is less than the Hölder exponent characterizing the singularity (*critical order*). Instead, in the singular points, the LFD (Eq. (4)) is generally zero or infinite. It assumes a finite value if and only if the order α of derivation is exactly equal to the Hölder exponent of the graph. For instance, in the case of the well-known devil's staircase graph (Fig. 2b) the LFD of order $\alpha = \log 2/\log 3$ (i.e. equal to the dimension of the underlying middle-third Cantor set) is zero everywhere except in the singular points where it is finite (see Section 7).

Starting from Eq. (4), Kolwankar defined, for functions of several variables, the directional LFD [20]. As a particular case, in the following we will use the partial LFD. Considering, for instance, the function $f(x, y, z)$, its partial LFD with respect to x reads:

$$\mathrm{D}_x^\alpha f(x, y, z) = \lim_{t\to x} \frac{\mathrm{d}^\alpha[f(t, y, z) - f(x, y, z)]}{[\mathrm{d}(t - x)]^\alpha}. \tag{5}$$

By means of the partial LFD, the fractal differential equations of kinematics and statics have been obtained in [1]. The displacement vector field $\{\eta\} = (u, v, w)^{\mathrm{T}}$ maintains the integer dimension of length. The noninteger dimensions of the fractal strain components $\{\varepsilon^*\} = (\varepsilon_x^*, \varepsilon_y^*, \varepsilon_z^*, \gamma_{xy}^*, \gamma_{xz}^*, \gamma_{yz}^*)^{\mathrm{T}}$ are: $[\mathrm{L}]^{d_\varepsilon}$. Therefore, they can be obtained by fractional differentiation of the displacement vector $\{\eta\}$, according to the definition of LFD outlined above. The fractional differential operator $[\partial^\alpha]$ can thus be introduced, where the order of differentiation is $\alpha = 1 - d_\varepsilon$. Thereby, the kinematic equations for the fractal medium can be written, in the usual vector notation, as:

$$\{\varepsilon^*\} = [\partial^\alpha]\{\eta\}, \tag{6}$$

where $[\partial^\alpha]$ explicitly reads:

$$[\partial^\alpha] = \begin{bmatrix} \mathrm{D}_x^\alpha & 0 & 0 \\ 0 & \mathrm{D}_y^\alpha & 0 \\ 0 & 0 & \mathrm{D}_z^\alpha \\ \mathrm{D}_y^\alpha & \mathrm{D}_x^\alpha & 0 \\ \mathrm{D}_z^\alpha & 0 & \mathrm{D}_x^\alpha \\ 0 & \mathrm{D}_z^\alpha & \mathrm{D}_y^\alpha \end{bmatrix}. \tag{7}$$

Eq. (6) represents also a *local* definition of the fractal strain, as compared to the mean-field one provided by renormalization group (see Fig. 2). Classical nondimensional strain is obtained when $\alpha = 1$ $(d_\varepsilon = 0)$. Instead, when $\alpha = 0$, strain is no longer homogeneously diffused and reduces to localized displacement discontinuities. The intermediate situations are described by generic values of α.

The static equations link the fractal stress vector $\{\sigma^*\} = (\sigma_x^*, \sigma_y^*, \sigma_z^*, \tau_{xy}^*, \tau_{xz}^*, \tau_{yz}^*)^{\mathrm{T}}$ to the vector of the body forces $\{F^*\} = (F_x^*, F_y^*, F_z^*)^{\mathrm{T}}$, which assumes noninteger dimensions according to the fractal dimension of the deformable subset Ω^*, $[\mathrm{F}][\mathrm{L}]^{-(3-d_\omega)}$. On the other hand, the dimensions of the fractal stress are $[\mathrm{F}][\mathrm{L}]^{-(2-d_\sigma)}$. Therefore, the fractal equilibrium equations can be written, in the vector notation, as:

$$[\partial^\alpha]^{\mathrm{T}}\{\sigma^*\} = -\{F^*\}, \tag{8}$$

where the static fractional differential operator $[\partial^\alpha]^{\mathrm{T}}$ is the transposed of the kinematic fractional differential operator $[\partial^\alpha]$. It is worth to observe that the fractional order of differentiation of the static operator in the fractal medium is $\alpha = 1 - d_\varepsilon$, the same as that of the kinematic operator, see Eq. (5). This remarkable result is due to the fundamental relation among the exponents (Eq. (2)), and represents the *Duality Principle for Fractal Media* [1]. Finally, equivalence at the boundary of the body requires that the stress vector coincides with the applied fractal boundary tractions $\{p^*\}$ (with physical dimensions $[\mathrm{F}][\mathrm{L}]^{-(2-d_\sigma)}$):

$$[N]^{\mathrm{T}}\{\sigma^*\} = \{p^*\}. \tag{9}$$

The definition of the normal vector on a fractal boundary is still a challenging mathematical problem. For lacunar fractals, $[N]^{\mathrm{T}}$ can be defined at any dense point of the boundary as the cosine matrix of the outward normal to the boundary of the initiator of the fractal body. In the case of bodies whose border is represented by an invasive fractal, the impossibility of establishing a pointwise definition of the normal vector can be overcome proving directly the extensions of the vector analysis theorems to domains with fractal boundaries. This can be done by using the notion of *oriented Iterated Function Systems* and directional pseudo-measures [21].

4. Principle of Virtual Work and linear elastic constitutive law for fractal media

The Principle of Virtual Work is the fundamental identity of solid mechanics. It affirms the equality between the *virtual external work* (done by body forces and boundary tractions) and the *virtual internal work* (done by internal stresses). As is well known, the proof of the principle requires the application of the Gauss–Green Theorem. Some attempts of extending the Gauss–Green Theorem to fractal domains have been discussed in our previous paper [1].

Following Kolwankar and Gangal [17,18], the formula of *local fractional integration by parts* can be written, in one dimension, as:

$$[f(x)g(x)]_a^b = \{I^\alpha[g(x)\mathrm{D}^\alpha f(x)]\}_a^b + \{I^\alpha[f(x)\mathrm{D}^\alpha g(x)]\}_a^b, \tag{10}$$

where both integrals and derivatives are fractional. Therefore, dimensional homogeneity is guaranteed by Eq. (10). Note, however, that the above formula is restricted to functions possessing the same Hölder exponent α, and that the order of integration coincides with the order of derivation. Consider now two arbitrary functions $f(x,y,z)$ and $g(x,y,z)$, defined in a 3D fractal domain Ω^*, with the same critical order α. The general formula of local fractional integration by parts has been obtained by the authors [1] as:

$$\{I^{(\beta-\alpha)}[gf]\}_{\Gamma^*} = \{I^\beta[g\mathrm{D}^\alpha f]\}_{\Omega^*} + \{I^\beta[f\mathrm{D}^\alpha g]\}_{\Omega^*}, \tag{11}$$

where Γ^* is the boundary of the 3D domain Ω^*. This result extends the Gauss–Green Theorem to 3D fractal domains. Based on Eq. (11), the Principle of Virtual Work for fractal media has been demonstrated [1]. It reads:

$$\int_{\Omega^*(3-d_\omega)} \{F_A^*\}^{\mathrm{T}}\{\eta_B\}\,\mathrm{d}\Omega^* + \int_{\Gamma^*(2-d_\sigma)} \{p_A^*\}^{\mathrm{T}}\{\eta_B\}\,\mathrm{d}\Gamma^* = \int_{\Omega^*(3-d_\omega)} \{\sigma_A^*\}^{\mathrm{T}}\{\varepsilon_B^*\}\,\mathrm{d}\Omega^*. \tag{12}$$

Both sides of Eq. (12) possess the dimensions of work ([F][L]), since the operators are *fractal integrals* defined upon fractal domains. The external work may be done by fractal body forces $\{F^*\}$ and/or by fractal tractions $\{p^*\}$ acting upon the boundary Γ^* of the body. The internal work of deformation is defined as $\mathrm{d}W^* = \{\sigma^*\}^{\mathrm{T}}\{\mathrm{d}\varepsilon^*\}$, with fractional dimensions $[\mathrm{F}][\mathrm{L}]^{-(2-d_\omega)}$. If the (initial) loading process is conservative (no dissipation occurs in the material), and stress is a univocal function of strain, $\mathrm{d}W^*$ represents a strain-energy function. In this case, a *fractal elastic potential* ϕ^* (function of the fractal strain $\{\varepsilon^*\}$) can be introduced as an exact differential:

$$\mathrm{d}\phi^* = \{\sigma^*\}^{\mathrm{T}}\{\mathrm{d}\varepsilon^*\}. \tag{13}$$

Accordingly, the components of the fractal stress vector $\{\sigma^*\}$ can be obtained by derivation of the fractal elastic potential ϕ^*:

$$\sigma_i^* = \frac{\partial \phi^*}{\partial \varepsilon_i^*}. \tag{14}$$

Note that these are canonical first-order partial derivatives in the space of the fractal strains $\{\varepsilon^*\}$. Therefore, the same arguments of classical mathematical elasticity can be followed. Performing the Taylor expansion around the undeformed state, and neglecting higher-order derivatives in the space of the $\{\varepsilon^*\}$, the following quadratic form is easily obtained:

$$\phi^* = \tfrac{1}{2}\{\varepsilon^*\}^{\mathrm{T}}[\boldsymbol{H}^*]\{\varepsilon^*\}, \tag{15}$$

where $[\boldsymbol{H}^*]$ is the Hessian matrix of the fractal elastic potential. The above formula shows that ϕ^* is always a positive quantity (i.e. positive mechanical work is needed to increase the deformations), provided that $[\boldsymbol{H}^*]$ is positively defined. Simple dimensional arguments show that the noninteger physical dimensions of $[\boldsymbol{H}^*]$ are:

$$\dim[\boldsymbol{H}^*] = [\mathrm{F}][\mathrm{L}]^{-(2+d_\varepsilon-d_\sigma)}. \tag{16}$$

Thus, $[\boldsymbol{H}^*]$ depends on both the dimensions of stress and strain. Each term in $[\boldsymbol{H}^*]$ is defined as the second-order partial derivative of the elastic potential by the corresponding fractal strains:

$$\boldsymbol{H}^*_{ij} = \frac{\partial^2\phi^*}{\partial\varepsilon^*_i\,\partial\varepsilon^*_j}. \tag{17}$$

From Eqs. (14) and (15), the *linear elastic constitutive law for fractal media* is provided as:

$$\{\sigma^*\} = [\boldsymbol{H}^*]\{\varepsilon^*\}. \tag{18}$$

By means of Eq. (18), the fractal elastic potential can be also written as a bilinear function of the fractal stress and of the fractal strain:

$$\phi^* = \tfrac{1}{2}\{\sigma^*\}^{\mathrm{T}}\{\varepsilon^*\}. \tag{19}$$

In a series of previous papers [4,5], the authors explained that, since the fractal mechanical quantities (i.e., $[\sigma^*]$, $[\varepsilon^*]$, W^*) are scale-invariant, the corresponding nominal quantities (i.e., nominal stress, strain and fracture energy) undergo size-effects depending on the fractal dimension of their counterparts. For instance, the well-known decrease of tensile strength with increasing structural size was successfully explained [4] by applying the renormalization group procedure to the fractal stress $\{\sigma^*\}$.

Following the same arguments, the anomalous dimensions of the fractal elastic constants $[\boldsymbol{H}^*]$ (see Eq. (16)) can explain the size-effects on the nominal elastic constants which, although less relevant than those on strength, have been detected by experimental tests [22]. The nominal elastic constants, relating nominal stresses and strains, have the same dimensions of stress, i.e., $[\mathrm{F}][\mathrm{L}]^{-2}$. This is the case, for instance, of the Young's modulus E and of the shear modulus G. Recalling the scaling laws for nominal stress and nominal strain [1,4], one obtains:

$$[\boldsymbol{H}] = \frac{[\sigma]}{[\varepsilon]} \sim \frac{\{\sigma^*\}b^{-d_\sigma}}{\{\varepsilon^*\}b^{-d_\varepsilon}} \sim [\boldsymbol{H}^*]b^{d_\varepsilon-d_\sigma}, \tag{20}$$

which represents the scaling law of the nominal elastic constants. In the logarithmic form:

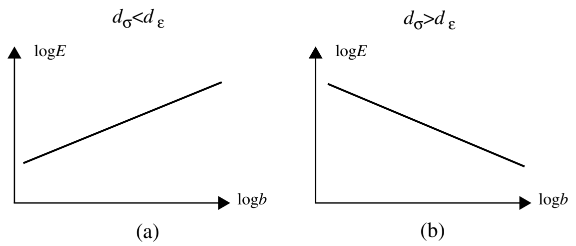

Fig. 4. Size-effects on the nominal elastic modulus E according to the physical dimensions of the fractal elastic constants.

$$\log[\boldsymbol{H}] = \log[\boldsymbol{H}^*] - (d_\sigma - d_\varepsilon)\log b. \tag{21}$$

The slope of the scaling law is thus equal to: $-(d_\sigma - d_\varepsilon)$. Therefore, depending on the difference $(d_\sigma - d_\varepsilon)$, the nominal elastic constants can be subjected to positive or negative size-effects (Fig. 4). Since it is physically plausible to state that in the elastic stage, before major strain localization, d_σ is larger than d_ε, the slope is negative, that is, the nominal Young's modulus decreases as the size of the body increases. For instance, in the case of a stretched rod with the same degree of cantorian fractality for the deformation field and for the cross-sections, one obtains:

$$d_\varepsilon = 0.36, \tag{22a}$$

$$d_\sigma = 2d_\varepsilon = 0.72, \tag{22b}$$

which implies, according to the fundamental relation (Eq. (2)): $d_\omega = d_\varepsilon + d_\sigma = 3d_\varepsilon = 1.08$. Thereby, the nominal Young's modulus E of the bar scales with size according to:

$$E \sim b^{-0.36}, \tag{23}$$

that is, deformability slightly increases with the size of the body (Fig. 4b), as experimentally detected for porous materials [22].

5. Direct Lamé formulation of the fractal elastic problem

If the fractal constitutive law is inserted into the fractal static equations (Eq. (8)), one obtains:

$$[\partial^\alpha]^{\mathrm{T}}[\boldsymbol{H}^*]\{\varepsilon^*\} = -\{F^*\}. \tag{24}$$

By using the fractal kinematic equations (Eq. (6)), the so-called Lamé equation is provided in the operatorial form:

$$\left([\partial^\alpha]^{\mathrm{T}}[\boldsymbol{H}^*][\partial^\alpha]\right)\{\eta\} = -\{F^*\}, \tag{25}$$

where the fractal Lamé operator $[L^*]_{3\times3}$ can be defined as:

$$[L^*] = [\partial^\alpha]^{\mathrm{T}}[\boldsymbol{H}^*][\partial^\alpha]. \quad (26)$$

Note that the dimensions of the fractal operator $[L^*]$ depend on the Hausdorff dimensions Δ_ω of the fractal body but are independent of the fractional dimensions of stress and strain:

$$\dim[L^*] = [\mathrm{F}][\mathrm{L}]^{-(4-d_\omega)}. \quad (27)$$

The conditions of equivalence on the boundary of the medium (Eq. (9)) can be expressed, as a function of the displacement vector, in the following manner:

$$\left([N]^{\mathrm{T}}[\boldsymbol{H}^*][\partial^\alpha]\right)\{\eta\} = \{p^*\}, \quad (28)$$

where $[N]^{\mathrm{T}}$ is the cosine matrix on the boundary, exactly corresponding to the fractional derivatives in $[\partial^\alpha]^{\mathrm{T}}$, in the spirit of the fractal Gauss–Green Theorem. The fractal operator $[L_0^*]_{3\times 3}$ is thus defined as:

$$[L_0^*] = [N]^{\mathrm{T}}[\boldsymbol{H}^*][\partial^\alpha]. \quad (29)$$

It is easy to show that the dimensions of the fractal operator $[L_0^*]$ are $[\mathrm{F}][\mathrm{L}]^{-(3-d_\sigma)}$, i.e., they only depend on those of the fractal stress. In conclusion, the problem of fractal elastostatics can be summarized, in terms of the displacements, as follows:

$$[L^*]\{\eta\} = -\{F^*\}, \quad \forall P \in \Omega^*; \quad (30a)$$

$$[L_0^*]\{\eta\} = \{p^*\}, \quad \forall P \in \Gamma_1^*; \quad (30b)$$

$$\{\eta\} = \{\eta_0\}, \quad \forall P \in \Gamma_2^*, \quad (30c)$$

where the boundary has been split, as usually, in a region where the tractions are imposed (Γ_1) and a region with prescribed displacements (Γ_2).

The equations of fractal elastodynamics are a straightforward extension of the static case. For instance, the equation of the free vibrations can be obtained from Eq. (30a) by simply substituting the vector of the fractal body forces $\{F^*\}$ with the (fractal) inertial forces:

$$-[\rho^*]\frac{\partial}{\partial t^2}\{\eta\}, \quad (31)$$

where $[\rho^*]$ is the density of the fractal medium, with dimensions $[\mathrm{M}][\mathrm{L}]^{-(3-d_\omega)}$. For lacunar fractal sets in the 3D space (e.g. Menger sponges), the canonical density ρ is scale-dependent and tends to zero for the largest sizes [1]. Instead, the *fractal density* ρ^* is a scale-invariant quantity and controls the fractal vibrational states.

6. Variational formulation and discretization of the fractal elastic problem

The total potential energy of a fractally deformable elastic body can be defined as the difference between the strain energy stored in the body minus the external work of the body forces $\{F^*\}$ and of the boundary tractions $\{p^*\}$:

$$W(\eta) = \int_{\Omega^*(3-d_\omega)} \phi^*\,\mathrm{d}\Omega^* - \left(\int_{\Omega^*(3-d_\omega)} \{F^*\}^{\mathrm{T}}\{\eta\}\,\mathrm{d}\Omega^* + \int_{\Gamma^*(2-d_\sigma)} \{p^*\}^{\mathrm{T}}\{\eta\}\,\mathrm{d}\Gamma^*\right). \quad (32)$$

This functional, defined in the space of the admissible displacement and strain fields, attains its minimum value in correspondence of the (unique) solution of the elastic problem. This can be demonstrated by following the same arguments established for the canonical quantities [23], i.e., by considering an arbitrary kinematically admissible field $\{\tilde{\eta}\}$, $\{\tilde{\varepsilon}^*\}$, obtained by adding, to the fields $\{\eta\}$, $\{\varepsilon^*\}$, which are solution of the elastic problem (Eqs. (30)), a generic increment $\{\Delta\eta\}$, $\{\Delta\varepsilon^*\}$, and evaluating the functional $W(\tilde{\eta})$. If the Principle of Virtual Work (Eq. (12)) is applied to the statically admissible stress-force field (defined by Eqs. (8) and (9)) working by the incremental kinematic fields $(\{\Delta\eta\}, \{\Delta\varepsilon^*\})$, using the Betti's Reciprocity Theorem one easily obtains:

$$W(\tilde{\eta}) - W(\eta) = \int_{\Omega^*(3-d_\omega)} \phi^*(\Delta\varepsilon^*)\,\mathrm{d}\Omega^*. \quad (33)$$

Recalling that the fractal elastic potential is a quadratic positive form, the above relation implies that the total potential energy related to the displacement field $\{\eta\}$, solution of the elastic problem, is the minimum with respect to any other admissible field $\{\eta + \Delta\eta\}$:

$$W(\eta) = \text{minimum}. \quad (34)$$

The above variational formulation is thus equivalent to the direct formulation of the elastic problem (Eqs. (30)) and represents the starting point of the Finite Element formulation. In the classical Ritz–Galerkin approach, the stationarity of the energy functional $W(\eta)$ is sought within a subspace of finite dimensions subtended by a series of linearly independent assigned functions. The problem is thus discretized, since the unknown displacement field $\{\eta\}$ is expressed as the sum of assigned independent functions $\{\eta_i\}$, $i = 1, 2, \ldots, (g \times n)$:

$$\{\eta\} = \sum_{i=1}^{(g\times n)} \beta_i\{\eta_i\}, \quad (35)$$

where n represents the number of nodes where the displacement has to be determined, each with g degrees of freedom. Note, incidentally, that the physical dimension of the functions $\{\eta_i\}$ is that of length. By inserting the above relation in the definition of the total potential energy, the following functional is obtained:

$$W(\beta) = \tfrac{1}{2}\{\beta\}^{\mathrm{T}}[\boldsymbol{L}]\{\beta\} - \{\beta\}^{\mathrm{T}}[\boldsymbol{F}], \tag{36}$$

where $\{\beta\}$ is the vector of the $(g \times n)$ unknown coefficients of the linear combination (Eq. (35)). The $(g \times n) \times (g \times n)$ elements of the matrix $[\boldsymbol{L}]$ are given by the following fractal integrals:

$$\boldsymbol{L}_{ij} = -\int_{\Omega^*(3-d_\omega)} \{\eta_i\}^{\mathrm{T}}[L^*]\{\eta_j\}\,\mathrm{d}\Omega^* + \int_{\Gamma^*(2-d_\sigma)} \{\eta_i\}^{\mathrm{T}}[L_0^*]\{\eta_j\}\,\mathrm{d}\Gamma^*. \tag{37}$$

Note that the matrix $[\boldsymbol{L}]$ is symmetrical due to Betti's Theorem. The $(g \times n)$ elements of the vector $[\boldsymbol{F}]$ are given by the following fractal integrals:

$$\boldsymbol{F}_{ij} = -\int_{\Omega^*(3-d_\omega)} \{\eta_i\}^{\mathrm{T}}\{F^*\}\,\mathrm{d}\Omega^* + \int_{\Gamma^*(2-d_\sigma)} \{\eta_i\}^{\mathrm{T}}\{p^*\}\,\mathrm{d}\Gamma^*. \tag{38}$$

Minimization of the above energy functional requires the canonical first-order derivatives of Eq. (36), with respect to each coefficient β_i, to be equal to zero. The result is the classical system of $(g \times n)$ linear algebraic equations in terms of $(g \times n)$ unknown coefficients β_i. Synthetically, it reads:

$$[\boldsymbol{L}]\{\beta\} = [\boldsymbol{F}]. \tag{39}$$

By using, as interpolating functions, the fractal splines that will be introduced in the next section, the physical meaning of the matrix $[\boldsymbol{L}]$ is that of a stiffness matrix. In a straightforward dual manner, the complementary elastic potential—function of the fractal stresses—can be introduced. The existence of such potential yields to the definition of the compliance matrix $[\boldsymbol{C}]$, with the usual mechanical significance.

7. Fractal interpolation by devils staircase splines

We can choose as independent functions $\{\eta_i\}$ in the linear combination (Eq. (35)) a family of functions that, henceforth, we will call *fractal splines*. These are defined only over subdomains (finite elements) of the whole fractal domain Ω^*, and their value is equal to one in correspondence of the pertinent node i and to zero in all the other nodes within their subdomain of definition (see Fig. 7a). Since their linear combination gives the singular displacement field, the physical dimension of the fractal splines must be that of a length [L] and, at the same time, they must be functions of the *devil's staircase* kind, i.e. continuous functions everywhere constant except on a subset with null Lebesgue measure. For our purposes, the most important aspect related to the fractal splines is their ability to model the singular displacement fields and to provide, by local fractional derivation, the associated fractal strain fields.

Since the fractal splines in the ID case (for finite elements with two nodes) can be built by connecting two devil's staircases (Fig. 7a), it is worthwhile describing more in details this archetype function, already presented in Section 2 and plotted in Fig. 2b. Consider a middle-third Cantor set, with dimension $\Delta = \log 2/\log 3 = 0.6309$. This is a well-known lacunar set with null Lebesgue measure. In order to construct a devil's staircase, let us consider, at the first iteration, the unitary segment with unitary mass (Fig. 5a). At the second iteration, the mass is concentrated over the first and last third of the bar, and so on. At the nth iteration, there will be $N = 2^n$ small segments each with length $l_i = 3^{-n}$ and mass $\mu_i = 2^{-n}$. The mass μ_i of the generic segment, as a function of its length l_i, is equal to $\mu_i = l_i^\alpha$, whereas its density ρ_i is:

$$\rho_i = \frac{\mu_i}{l_i} = l_i^{\alpha-1}, \tag{40}$$

where $\alpha = \log 2/\log 3$ (i.e. the fractal dimension of the underlying Cantor set) is called the *scaling exponent*. Notice that, while the mass μ_i tends to zero as $n \to \infty$, the density ρ_i diverges because $\alpha < 1$. Let us now calculate the graph of the cumulative mass $S(x)$ as a function of the abscissa x $(0 < x < 1)$:

$$S(x) = \int_0^x \rho(t)\,\mathrm{d}t = \int_0^x \mathrm{d}\mu(t). \tag{41}$$

The diagram $S(x)$ is constant over the voids of the Cantor set, where the density is equal to zero. Since the sum of the voids covers the whole interval $[0, 1]$, one could wrongly imagine that $S(x) = 0$. On the contrary, $S(x)$ grows from zero to one (Fig. 5b) through infinitesimal jumps localized in correspondence of the (infinite) points belonging to the Cantor set. The graph of

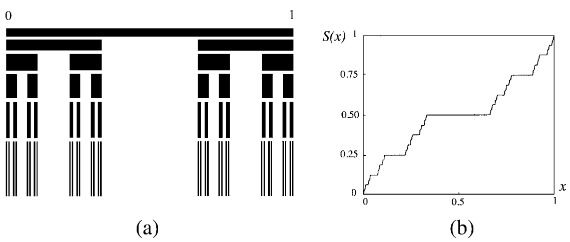

Fig. 5. Construction of the devil's staircase function $S(x)$ supported by a middle-third Cantor set (a) and graph of the function $S(x)$ over the interval $[0, 1]$ (b).

$S(x)$ is the archetype of the *devil's staircase* functions, which are self-affine, monotone, nondifferentiable functions. The singularities of $S(x)$ can be characterized by the Hölder exponent $\alpha < 1$. For a generic function $f(x)$, analytical in x_0, one obtains $\alpha(x_0) = \infty$. Instead, if $\alpha(x_0) < 1$, the function presents a singularity of order α in x_0 (i.e., the first derivative is not defined). It can be easily demonstrated that the Hölder exponent of the function $S(x)$, in all the points belonging to the Cantor set, is exactly equal to the fractal dimension $\Delta = 0.6309$ of the fractal support where it grows. In other words, the devil's staircase is a function which is almost everywhere constant, except on a subset of points with dimension Δ, where it is singular of order $\alpha = \Delta$.

It is important to notice that the graph of function $S(x)$ is not a fractal in the strict sense, since its total length is measurable and equal to 2. Indeed, its first derivative is zero everywhere and is not defined on all the points of the Cantor set. Moreover, as Fig. 5b clearly shows, $S(x)$ is a self-affine function, since:

$$S(\lambda x) = \lambda^{\alpha} S(x), \quad \lambda = (1/3)^n, \ n \in N. \tag{42}$$

Let us now discuss the fractional differentiability of $S(x)$. Tricot [24] demonstrated that, for any self-affine function with fractal dimension equal to 2-H, the maximum order of fractional differentiability is equal to H. The classical fractional derivative (with the lower extreme of integration coincident with the origin) of the devil's staircase was extensively studied by Schellnhuber and Seyler [25]. For the calculations, the devil's staircase can be conveniently approximated with a polygonal family of functions $S_n(x)$:

$$S_n(x) = \sum_{i=0}^{2^{n+1}-2} (-1)^i \left(\frac{3}{2}\right)^n H_{x_n(i)}(x)[x - x_n(i)], \tag{43}$$

where $H_{x_0}(x)$ is the unit step function with discontinuity in x_0 and $x_n(i)$ are the x-coordinates of the extremes of the segments of the nth step in the iterative procedure leading to the construction of the Cantor set. It can be proved that the sequence $S_n(x)$ converges uniformly to $S_n(x)$ for $n \to \infty$. The Riemann–Liouville fractional derivative of the generic nth term $S_n(x)$ can be calculated [25] to give:

$$\frac{\mathrm{d}^{\alpha}[S_n(x)]}{[\mathrm{d}x]^{\alpha}} = \sum_{i=0}^{2^{n+1}-2} (-1)^i \left(\frac{3}{2}\right)^n \frac{H_{x_n(i)}(x)}{\Gamma(2-\alpha)} [x - x_n(i)]^{1-\alpha}. \tag{44}$$

Schellnhuber and Seyler [25] demonstrated the uniform convergence of the sequence of fractional derivatives (44) for $\alpha < \log 2/\log 3$, thus obtaining the fractional derivative of order α of $S(x)$. For $\alpha \geqslant \log 2/\log 3$, the sequence (44) diverges. Thereby, the devil's staircase, although nondifferentiable in the classical sense, admits continuous fractional derivatives (Fig. 6), provided that its order α is lower than Δ.

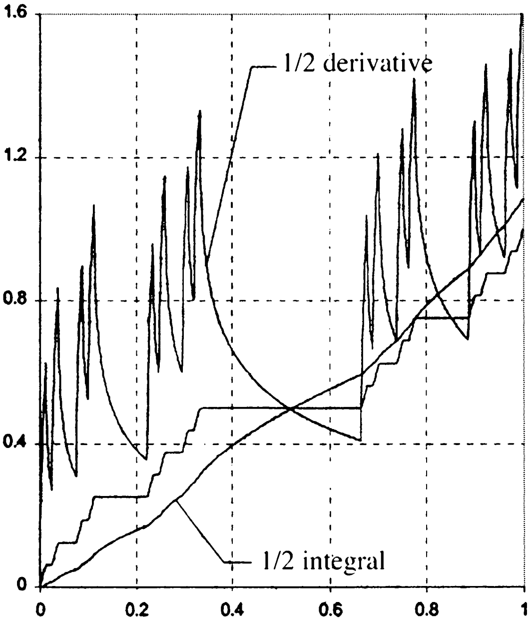

Fig. 6. Graph of the 1/2-order fractional derivative and of the 1/2-order fractional integral of the devil's staircase function $S(x)$.

For our model, we are mainly interested in the local fractional derivative (LFD) of the devil's staircase. Based on the classical fractional derivative, it is possible to demonstrate that the maximum order of (classical) fractional derivative $\alpha_{\max}$ coincides with the critical order of the local fractional derivative. For instance, the LFD of the devil's staircase in the origin is given by:

$$\mathrm{D}^{\alpha}[S(0)] \cong \lim_{n\to\infty} \left\{ \frac{\mathrm{d}^{\alpha}[S_n(x)]}{[\mathrm{d}x]^{\alpha}} \right\}_{x=1/3^n} = \frac{1}{\Gamma(2-\alpha)} \lim_{n\to\infty} \left(\frac{3^{\alpha}}{2}\right)^n. \tag{45}$$

Therefore, the LFD of $S(x)$ in the origin is zero if $\alpha < \Delta$, does not exist if $\alpha > \Delta$, and attains a finite value if and only if $\alpha = \Delta$. The same occurs in all the singular points of the devil's staircase graph while, where the graph is flat, the LFD is zero. In other words, we have shown that the devil's staircase can be differentiated by the LFD of order equal to the dimension of the fractal support where it grows.

The conclusions obtained for the function $S(x)$ can be generalized to the fractal splines and to their linear combination (see Fig. 7). Therefore, these functions can be considered as a suitable family of functions to interpolate the singular displacement fields which arise in bodies deformable on fractal subsets. Moreover,

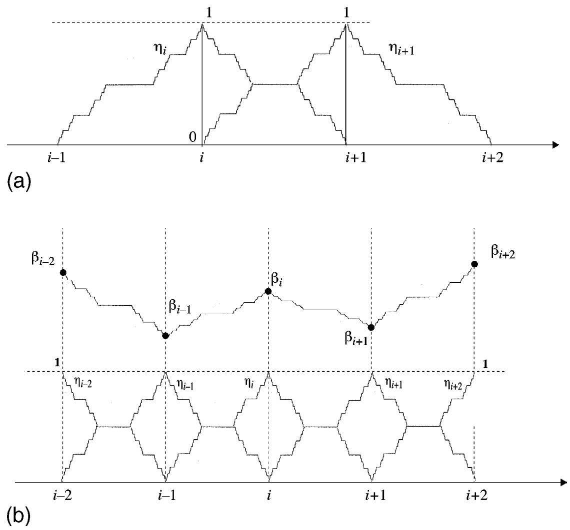

Fig. 7. Hat-like devil's splines (a) and interpolation of a displacement field by a quasi-orthonormal and complete set of splines (b).

choosing as the order of fractional derivation the fractal dimension Δ of the subset where the body can get deformed, the LFD can provide the associated discontinuous strain fields. On the other hand, it is evident how they represent a quasi-orthonormal and complete functional basis for numerical approximations (Fig. 7b).

Acknowledgements

Supports by the Italian Ministry of University and Scientific Research and by the EC-TMR Contract No. ERBFMRXCT 960062, are gratefully acknowledged.

References

[1] Carpinteri A, Chiaia B, Cornetti P. Static–kinematic duality and the principle of virtual work in the mechanics of fractal media. Comput Methods Appl Mech Eng 2001;190:6053–73.
[2] Cosserat E, Cosserat F. Theorie des Corps Déformables. Paris: Hermann; 1909.
[3] Aifantis EG. On the role of gradients in the localisation of deformation and fracture. Int J Eng Sci 1992;30:1279–99.
[4] Carpinteri A. Fractal nature of materials microstructure and size effects on apparent mechanical properties. Mech Mater 1994;18:89–101.
[5] Carpinteri A, Chiaia B, Cornetti P. On the mechanics of quasibrittle materials with a fractal microstructure. Eng Fract Mech 2003;70:2321–49.
[6] Borri-Brunetto M, Carpinteri A, Chiaia B. Scaling phenomena due to fractal contact in concrete and rock fractures. Int J Fract 1999;95:221–38.
[7] Radlinski AP, Radlinska EZ, Agamalian M, Wignall GD, Lindner P, Randl OG. Fractal geometry of rocks. Phys Rev Lett 1999;82:3078–81.
[8] Vallejo LE. Fractal analysis of the fabric changes in a consolidating clay. Eng Geol 1996;43:281–90.
[9] Herrmann HJ. Some new results on fracture. Physica A 1995;221:125–33.
[10] Herrmann HJ, Roux S. Statistical models for the fracture of disordered media. Amsterdam: North Holland; 1990.
[11] Meakin P. Stress distribution for a rigid fractal embedded in a two dimensional elastic medium. Phys Rev A 1987;36:325–31.
[12] Alexander S, Orbach R. Observation of fractons in silica aerogels. Europhys Lett 1988;6:245–50.
[13] Cornetti P. Fractals and fractional calculus in the mechanics of damaged solids. PhD thesis (in Italian), Politecnico di Torino, Torino, 1999.
[14] Kleiser T, Bocek M. The fractal nature of slip in crystals. Z MetaIlkd 1986;77:582–7.
[15] Poliakov AB, Herrmann HJ, Podladchikov YY, Roux S. Fractal plastic shearbands. Fractals 1995;2:567–81.
[16] Kolwankar KM, Gangal AD. Local fractional calculus: a calculus for fractal space–time. In: Fractals: theory and application in engineering. Delft: Springer; 1999.
[17] Kolwankar KM, Gangal AD. Fractional differentiability of nowhere differentiable functions and dimensions. Chaos 1996;6:505–23.
[18] Kolwankar KM, Gangal AD. Local fractional Fokker–Planck equation. Phys Rev Lett 1998;80:214–7.
[19] Oldham KB, Spanier J. The fractional calculus. New York: Academic Press; 1974.
[20] Kolwankar KM, Gangal AD. Local fractional derivatives and fractal functions of several variables. In: Proceedings of the Conference "Fractals in Engineering". Singapore: World Scientific; 1997.
[21] Giona M. Vector analysis on fractal curves. In: Proceedings of Fractals: Theory and Applications in Engineering. Delft, The Netherlands: Springer; 1999. p. 155–70.
[22] Duxbury PM. Breakdown of diluted and hierarchical systems. In: Herrmann HJ, Roux S, editors. Statistical models for the fracture of disordered media. Amsterdam: North Holland; 1990.
[23] Carpinteri A. Structural mechanics—a unified approach. London: E & FN Spon; 1997.
[24] Tricot C. Dérivation Fractionnaire et Dimension Fractale. Tech. Report 1532, CRM, Université de Montréal, 1988.
[25] Schellnhuber HJ, Seyler A. Fractional differentiation on devil's staircase. Physica A 1992;191:491–500.