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Short Communication

A disordered microstructure material model based on fractal geometry and fractional calculus

Alberto Carpinteri*, **Bernardino Chiaia, and Pietro Cornetti**

Politecnico di Torino, Department of Structural and Geotechnical Engineering, Torino, Italy

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Fractal patterns often arise in the failure process of materials with a disordered microstructure. It is shown that they are responsible of the size effects on the parameters characterizing the material behaviour in tensile tests (i.e. the strength, the fracture energy, and the critical displacement). Based on fractal geometry, a simple model of a generic disordered material is set. The physical quantities describing the stress-strain state of such fractal medium are pointed out. They show anomalous (non integer) physical dimensions. In terms of these fractal quantities, it is possible to define a fractal cohesive law, i.e. a constitutive law describing the tensile failure of an heterogeneous material, which is scale invariant. Then we propose new mathematical operators from fractional calculus to handle the fractal quantities previously introduced. In this way, the static and kinematic (fractional) differential equations of the model are pointed out. These equations form the basis of the mechanics of fractal media. In this framework, the principle of virtual work is also obtained.

1 Introduction

In solid mechanics, with the term *size effect* we mean the dependence of one or more material parameters on the size of the structure made by that material. In other words, we speak of size effect when geometrically similar structures show a different structural behavior. The first observations about size effect in solid mechanics date back to Galileo. For instance, in his "Discorsi e dimostrazioni matematiche intorno a due nuove scienze attenenti alla meccanica e i movimenti locali" (1638), he observed that the bones of small animals are more slender than the bones of big animals. In fact, increasing the size, the growth of the load prevails on the growth of the strength, since the first increases with the bulk, the latter with the area of the fracture surface. In the last century, fracture mechanics allowed a deeper insight in the size effect phenomenon. Nowadays, the most used model to describe damage localization in materials with disordered microstructure (also called quasi-brittle materials) is the *cohesive crack model*, introduced by Hillerborg et al. [1].

According to Hillerborg's model, the material is characterized by a stress-strain relationship (σ-ε), valid for the undamaged zones, and by a stress-crack opening displacement relationship (σ-w, the cohesive law), describing how the stress decreases from its maximum value σ_u to zero as the distance between the crack lips increases from zero to the critical displacement w_c. The area below the cohesive law represents the energy $\mathcal{G}_F$ spent to create the unit crack surface. The cohesive crack model is able to simulate tests where high stress gradients are present, e.g. tests on pre-notched specimens; in particular, it captures the ductile-brittle transition occurring by increasing the structural size. On the other hand, relevant scale effects are encountered also in uniaxial tensile tests on dog-bone shaped specimens [2, 3], where smaller stress gradients are present. In this case size effects, which should be ascribed to the material rather than to the stress-intensification, cannot be predicted by the cohesive crack model. In the following section, a scale-independent damage model is proposed which overcomes the drawbacks of the original cohesive model, assuming that damage occurs within a band where it is spread in a fractal way. The fractal nature of the damage process allows us to explain the size effects on tensile strength, fracture energy, and critical displacement and, particularly, the rising of the cohesive law tail observed in [3].

2 Damage mechanics of materials with heterogeneous microstructure

Let us start our investigation about materials with disordered microstructures analyzing the size effect on their tensile strength. Recent experimental results about porous concrete microstructure [4] led us to believe that a consistent modelling of damage in concrete can be achieved by assuming that the rarefied resisting sections in correspondence of the critical load can be

* Corresponding author, e-mail: alberto.carpinteri@polito.it

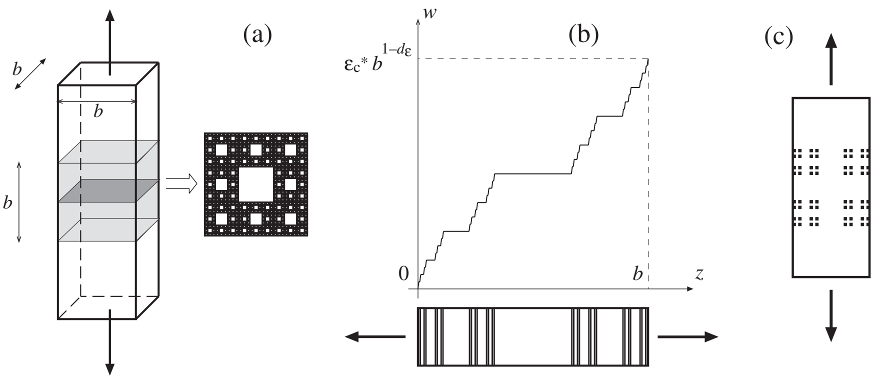

Fig. 1 Fractal localization: of the stress (a), of the strain (b), of the energy dissipation (c).

represented by stochastic lacunar fractal sets with dimension $2 - d_\sigma$ ($d_\sigma \geq 0$). From fractal geometry, we know that the area of lacunar sets is scale-dependent and tends to zero as the resolution increases. Finite measures can be obtained only with non-integer (fractal) dimensions. For the sake of simplicity, let us represent the specimen cross-section as a Sierpinski carpet built on the square of side b (Fig. 1a). The fractal dimension of this planar domain is 1.893 ($d_\sigma = 0.107$). The assumption of Euclidean domain characterizing the classical continuum theory states that the maximum load F is given by the product of the strength σ_u times the nominal area $A_0 = b^2$, whereas, in our model, F equals the product of the Hausdorff measure $A^* = b^{2-d_\sigma}$ of the Sierpinski carpet times the *fractal tensile strength* σ_u^* [5]:

$$F = \sigma_u A_0 = \sigma_u^* A^*, \tag{1}$$

where σ_u^* presents the anomalous physical dimensions $[F][L]^{-(2-d_\sigma)}$. The fractal tensile strength is the true material constant, i.e., it is scale-invariant. From eq. (1) we obtain the scaling law for tensile strength:

$$\sigma_u = \sigma_u^* b^{-d_\sigma}, \tag{2}$$

i.e. a power law with negative exponent $-d_\sigma$. Eq. (2) represents the negative size effect on tensile strength, experimentally revealed by several authors. Experimental and theoretical results allow us to affirm that d_σ can vary between the lower limit 0 – canonical dimensions for σ_u^* and absence of size effect on tensile strength – and the upper limit 1/2 – σ_u^* with the dimensions of a stress-intensity factor and maximum size effect on tensile strength (as in the case of LEFM).

Turning now our attention from a single cross-section to the whole damage zone, it can be noticed that damage is not localized onto a single section but is spread over a finite band where the damage distribution often presents fractal patterns. This is quite common in material science. For instance, in some metals, the so-called slip-lines develop with typical fractal patterns. Also fractal crack networks develop in dry clay or in old paintings under tensile stresses due to shrinkage. Thus, as representative of the damaged band, consider now the simplest structure, a bar subjected to tension, where, at the maximum load, dilation strain tends to concentrate into different softening regions, while the rest of the body undergoes elastic unloading. If, for the sake of simplicity, we assume that strain is localized onto cross-sections whose projections on the longitudinal axis are provided by a Cantor set, the displacement function at rupture can be represented (Fig. 1b) by a Cantor staircase graph (sometimes called devil's staircase). The strain defined in the classical manner is meaningless in the singular points, as it tends to diverge. This drawback can be overcome introducing a fractal strain. Let $1 - d_\varepsilon = 0.6391$ be, for instance, the fractal dimension of the lacunar projection of the damaged sections ($d_\varepsilon \geq 0$). According to the fractal measure of the damage line projection, the total elongation of the band at rupture must be given by the product of the Hausdorff measure $b^{(1-d_\varepsilon)}$ of the Cantor set times the *fractal critical strain* ε_c^*, while in the classical continuum theory it equals the product of the length b times the critical strain ε_c:

$$w_c = \varepsilon_c b = \varepsilon_c^* b^{(1-d_\varepsilon)}, \tag{3}$$

where ε_c^* has the anomalous physical dimension $[L]^{d_\varepsilon}$. The fractal critical strain is the true material constant, i.e., it is the only scale-invariant parameter governing the kinematics of the fractal band. On the other hand, eq. (3) states that the scaling of the critical displacement is described by a power law with positive exponent $1 - d_\varepsilon$. The fractional exponent d_ε is intimately related to the degree of disorder in the mesoscopic damage process. When d_ε varies from 0 to 1, the kinematical control parameter ε_c^* moves from the canonical critical strain ε_c – $[L]^0$ – to the critical crack opening displacement w_c – $[L]^1$. Therefore, when $d_\varepsilon = 0$ (diffused damage, ductile behavior), one obtains the classical response, i.e. collapse governed by the strain ε_c, independently of the bar length. In this case, continuum damage mechanics holds, and the critical displacement w_c is subjected to the maximum size effect ($w_c \sim b$). On the other hand, when $d_\varepsilon = 1$ (localization of damage onto a single section, brittle behavior) fracture mechanics holds and the collapse is governed by the critical displacement w_c, which is size-independent as in the cohesive model.

For what concerns the size effect upon the third parameter characterizing the cohesive law, i.e. the fracture energy $\mathcal{G}_F$, several experimental investigations have shown that $\mathcal{G}_F$ increases with the size of the specimen. This behavior can be explained by assuming that, after the peak load, the energy is dissipated inside the damage band, i.e. over the infinite lacunar sections where softening takes place (Fig. 1a,b). Generalizing eqs. (2) and (3) to the whole softening regime, we get $\sigma = \sigma^* b^{-d_\sigma}$ and $w = \varepsilon^* b^{(1-d_\varepsilon)}$. These relationships can be considered as changes of variables and applied to the integral definition of the fracture energy:

$$\mathcal{G}_F = \int_0^{w_c} \sigma \mathrm{d}w = b^{1-d_\varepsilon-d_\sigma} \int_0^{\varepsilon_c^*} \sigma^* \mathrm{d}\varepsilon^* = \mathcal{G}_F^* b^{1-d_\varepsilon-d_\sigma}. \tag{4}$$

Eq. (4) highlights the effect of the structural size on the fracture energy. On the other hand, since (Fig. 1c) the damage process takes place over an invasive fractal domain A^* (different from the lacunar one of eq. (1)) with a dimension $(2 + d_\mathcal{G})$ larger than 2 ($d_\mathcal{G} \geq 0$), we can also affirm that the total energy expenditure W is equal to [4]

$$W = \mathcal{G}_F A_0 = \mathcal{G}_F^* A^*, \tag{5}$$

where $\mathcal{G}_F^*$ is called the *fractal fracture energy* and presents the anomalous physical dimensions $[FL][L]^{-(2+d_\mathcal{G})}$ and, as well as σ_u^* and ε_c^*, it is scale-independent. Since $A_0 = b^2$ and $A^* = b^{2+d_\mathcal{G}}$, the value of $d_\mathcal{G}$ is linked to the values of d_σ and d_ε:

$$d_\sigma + d_\varepsilon + d_\mathcal{G} = 1, \tag{6}$$

where all the exponents are positive. While d_ε can get all the values inside the interval $[0, 1]$, d_σ and $d_\mathcal{G}$ tend to be comprised between 0 and 1/2 (brownian disorder). Eq. (6) states a strict restriction to the maximum degree of disorder, confirming that the sum of d_σ and $d_\mathcal{G}$ is always lower than 1, as previously asserted by Carpinteri through dimensional analysis arguments [5].

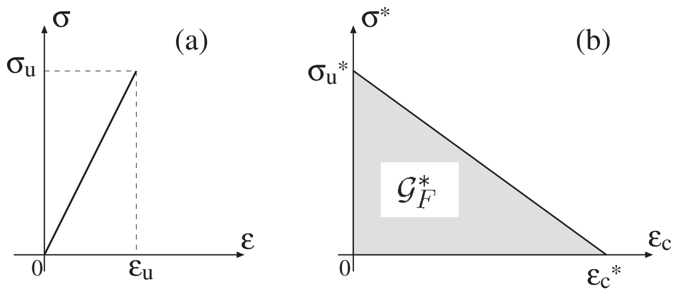

Fig. 2 Fractal cohesive model.

It is interesting to note how, from eq. (4), the fractal fracture energy $\mathcal{G}_F^*$ can be obtained as the area below the softening fractal stress-strain diagram (Fig. 2b). During the softening regime, i.e., when dissipation occurs, σ^* decreases from the maximum value σ_u^* to 0, while ε^* grows from 0 to ε_c^*. In the meantime, the non-damaged parts of the bar undergo elastic unloading (Fig. 2a). We call the σ^*-ε^* diagram the scale-independent or *fractal cohesive law*. Contrarily to the classical cohesive law, which is experimentally sensitive to the structural size, this curve should be an exclusive property of the material since it is able to capture the fractal nature of the damage process.

Recently, van Mier et al. [3] accurately performed tensile tests on dog-bone shaped concrete specimens over a wide scale range (1:32). They plotted the cohesive law for specimens of different sizes and found that, increasing the specimen size, the peak of the curve decreases whereas the tail rises. More in detail, w_c increases more rapidly than σ_u decreases, since, in the meantime, an increase of the area below the cohesive law, i.e. of the fracture energy, is observed. Thus, the fractal model consistently confirms the experimental trends of σ_u, $\mathcal{G}_F$, w_c.

The model has been applied to the data obtained by Carpinteri and Ferro [2,6] for tensile tests on dog-bone shaped concrete specimens (Fig. 3a) of various sizes under fixed boundary conditions. They interpreted the size effects on the tensile strength and the fracture energy by fractal geometry. Fitting the experimental results, they found the values $d_\sigma = 0.14$ and $d_\mathcal{G} = 0.38$. Some of the $\sigma - \varepsilon$ and the $\sigma - w$ diagrams are reported in Fig. 3b,c, where w is the displacement localized in the damage band, obtained by subtracting, from the total one, the displacement due to elastic and anelastic pre-peak deformation. Eq. (6) yields $d_\varepsilon = 0.48$, so that the fractal cohesive laws can be represented in Fig. 3d. As expected, all the curves related to the single sizes tend to merge in a unique, scale-independent cohesive law. The overlapping of the cohesive laws for the different sizes proves the soundness of the fractal approach in the interpretation of the size effects in concrete.

3 Fractional calculus, local fractional calculus, and fractal functions

The main characteristic of fractals is their irregularity over all the length scales. This irregularity is the reason of the non-integer dimensions of fractal sets and, unfortunately, it makes them very difficult to handle analytically since the usual calculus is

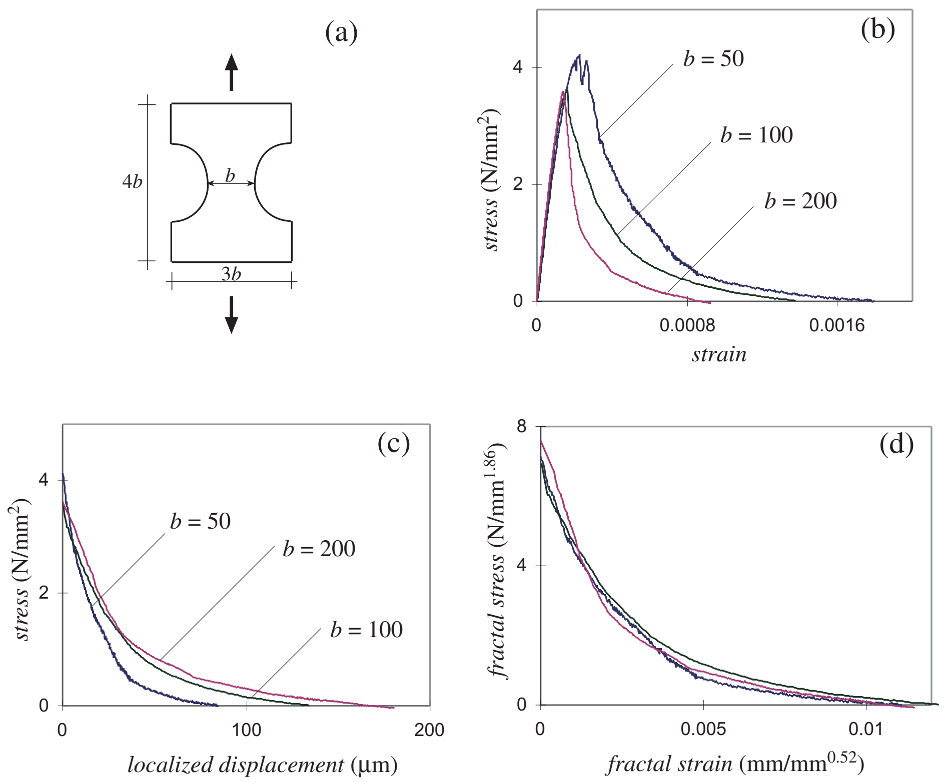

Fig. 3 Tensile tests over dog-bone shaped concrete specimens (a); stress versus strain plots (b), cohesive laws (c), fractal cohesive law (d). The specimen width b is expressed in millimetres.

inadequate to describe such structures and processes. Fractals are too irregular to have any smooth differentiable function defined on them. Fractal functions do not possess first order derivative at any point. Therefore it is argued that a new calculus should be developed which includes intrinsically a fractal structure [7]. Recently, Kolwankar [8], based on fractional calculus, defined new mathematical operators – the local fractional derivative and the fractal integral – that appear to be useful in the description of fractal processes. It is important to emphasize that, what seems to be really interesting in studying fractals via fractional calculus, are the non-integer physical dimensions that arise dealing with both fractional operators and fractal sets. Physically, this means to find the same scaling laws both from an analytic and a geometric point of view.

Let us start our analysis from the classical fractional calculus. While classical calculus treats integrals and derivatives of integer order, fractional calculus is the branch of mathematics that deals with the generalization of integrals and derivatives to all real (and even complex) orders. There are various definitions of fractional differintegral operators not necessarily equivalent to each other. A complete list of these definitions can be found in the fractional calculus treatises [9–12]. These definitions have different origin. The most frequently used definition of a fractional integral of order q ($q > 0$) is the Riemann-Liouville definition, which is a straightforward generalization to non-integer values of Cauchy formula for repeated integration:

$$\frac{\mathrm{d}^{-q} f(x)}{[\mathrm{d}(x-a)]^{-q}} = \frac{1}{\Gamma(q)} \int_a^x \frac{f(y)}{(x-y)^{1-q}} \, \mathrm{d}y. \tag{7}$$

From this formula, it appears logical to define the fractional derivative of order $n-1 < q < n$ (n integer) as the n-th integer derivative of the $(n-q)$-th fractional integral:

$$\frac{\mathrm{d}^{q} f(x)}{[\mathrm{d}(x-a)]^{q}} = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_a^x \frac{f(y)}{(x-y)^{q+1-n}} \, \mathrm{d}y. \tag{8}$$

Once these definitions are given, it is natural to write differential equations in terms of such quantities. In the last decade, many fractional differential equations have been proposed. They include fractional relaxation equations, wave equations, diffusion equations, etc. [13]. These equations are a generalization of the classical ones, the usual integer order time derivative being replaced by a fractional one. Anyway, to extend the classical solutions, one has to transform the whole initial value problem [14].

The use of the fractional differential operators is particularly meaningful since it yields a continuous transition between completely different models of the mathematical physics. Furthermore, it is worth mentioning that the application of the fractional relaxation to viscoelasticity [15] have led to solutions expressed by Mittag-Leffler type functions in good agreement with experimental results [16]. The fractional model is indeed able to describe processes that exhibit a *super slow* relaxation.

We want now to look for a link between fractional calculus and fractals. It can be noticed that, when q is not a positive integer, the fractional derivative (8) is a non-local operator since it depends on the lower integration limit a. The chain rule, Leibniz rule, composition law, and other properties have been studied for the fractional derivatives [9]. It is worthwhile to cite here the following scaling property (for $a = 0$):

$$\frac{\mathrm{d}^q f(bx)}{[\mathrm{d}x]^q} = b^q \frac{\mathrm{d}^q f(bx)}{[\mathrm{d}(bx)]^q}. \tag{9}$$

It means that the fractional differential operators are subjected to the same scaling power laws the quantities defined on fractal domains are subjected to (q being the fractal dimension). For the scaling property in the case $a \neq 0$, see [9].

More recently, another important result has been achieved concerning the maximum order of fractional differentiability for non-classical differentiable functions. Let us explain this property for two kinds of functions: the Weierstrass function, whose fractional differentiability was studied by Love et al. [17], and the Cantor staircase, whose fractional derivatives were obtained in [18]. The first one is continuous but nowhere differentiable. The singularities are locally characterized by the Hölder exponent, which is everywhere constant and equal to a certain value $0 < s < 1$. It is possible to prove that the graph of this function is fractal with a box-counting dimension equal to $2 - s$ and hence greater than 1. Although fractal, the Weierstrass function admits continuous fractional derivatives of order lower than s. Hence, there is a direct relationship between the fractal dimension of the graph and the maximum order of differentiability: the greater the fractal dimension, the lower the differentiability.

For what concerns the Cantor staircase function, we have already encountered it in Sect. 2. This kind of functions (Fig. 1b) can be obtained [19] as the integral of a constant mass density upon a lacunar fractal set belonging to the interval $[0, 1]$. The result is a monotonic function that grows on a fractal support; elsewhere it is constant. The devil's staircases are not fractal since they present a finite length; on the other hand, they have an infinite number of singular points characterized by a Hölder exponent equal to the fractal dimension of the support. Schellnhuber and Seyler [18] proved that the Cantor staircases admit continuous fractional derivatives of order lower than the fractal dimension of the set where they grow.

From a physical point of view, some efforts have been spent to apply space fractional differential equations to the study of phenomena involving fractal distributions in space. Here we can quote Giona and Roman [20], who proposed a fractional equation to describe diffusion on fractals, and Nonnenmacher [21], who showed that a class of Lévy type processes satisfies an integral equation of fractional order. This order is also the fractal dimension of the set visited by a random walker whose jump size distribution follows the given Lévy distribution.

Recently, a new notion called *local fractional derivative* (LFD) has been introduced with the motivation of studying the local properties of fractal structures and processes [22]. The LFD definition is obtained from (8) introducing two "corrections" in order to avoid some physically undesirable features of the classical definition. In fact, if one wishes to analyze the local behavior of a function, both the dependence on the lower limit a and the fact that adding a constant to a function yields to a different fractional derivative should be avoided. This can be obtained subtracting from the function the value of the function at the point where we want to study the local scaling property and choosing as the lower limit that point itself. Therefore, restricting our discussion to an order q comprised between 0 and 1, the LFD is defined as the following limit (if it exists and is finite):

$$D^q f(y) = \lim_{x \to y} \frac{\mathrm{d}^q [f(x) - f(y)]}{[\mathrm{d}(x-y)]^q}, \quad 0 < q \leq 1. \tag{10}$$

In [22] it has been shown that the Weierstrass function is locally fractionally differentiable up to a "critical order" α between 0 and 1. More precisely, the LFD is zero if the order is lower than α, does not exist if greater, while exists and is finite only if equal to α. Thus the LFD shows a behavior analogous to the Hausdorff measure of a fractal set. Furthermore, the critical order is strictly linked to the fractal properties of the function itself. In fact, Kolwankar and Gangal [22] showed that the critical order is equivalent to the local Hölder exponent (which depends, as we have seen, on the fractal dimension), by proving the following local fractional Taylor expansion of the function $f(x)$ of order $q < 1$ (for $q > 1$, see [23, 24]) for $x \to y$:

$$f(x) = f(y) + \frac{D^q f(y)}{\Gamma(q+1)} (x-y)^q + R_q(x-y), \tag{11}$$

where $R_q(x - y)$ is a remainder, negligible if compared with the other terms. Let us observe that the terms in the right hand side of eq. (11) are nontrivial and finite only if q is equal to the critical order α. Moreover, for $q = \alpha$, the fractional Taylor expansion (11) gives us the geometrical interpretation of the LFD. When q is set equal to unity, one obtains from (11) the equation of a tangent. All the curves passing through the same point y with the same first derivative have the same tangent. Analogously, all the curves with the same critical order α and the same D^α form an equivalence class modeled by x^α. This is how it is possible to generalize the geometric interpretation of derivatives in terms of "tangents".

The solution of the simple differential equation $\mathrm{d}f/\mathrm{d}x = 1_{[0,x]}$ gives the length of the interval $[0, x]$. The solution is nothing but the integral of the unit function. Wishing to extend this idea to the computation of the measure of fractal sets, it can be seen immediately that the fractional integral (7) does not work as it fails to be additive because of its non-trivial kernel. On the other hand, Kolwankar [25] proved that a fractional measure of a fractal set can be obtained through the inverse of the LFD defined as

$$ {}_aD_b^{-\alpha} f(x) = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i^*) \frac{\mathrm{d}^{-\alpha} 1_{\mathrm{d}x_i}(x)}{[\mathrm{d}(x_{i+1}-x_i)]^{-\alpha}}, \tag{12} $$

where $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$, $x_0 = a$, and $x_N = b$ provide a partition of the interval $[a, b]$ and x_i^* is some suitable point chosen in the subinterval $[x_i, x_{i+1}]$, while $1_{\mathrm{d}x_i}$ is the unit function defined on the same subinterval. Kolwankar called ${}_aD_b^{-\alpha} f(x)$ the *fractal integral* of order α of $f(x)$ over the interval $[a, b]$. The simple local fractional differential equation $D^\alpha f(x) = g(x)$ has not a finite solution when $g(x)$ is constant and $0 < \alpha < 1$. Interestingly, the solution exists if $g(x)$ has a fractal support whose Hausdorff dimension d is equal to the fractional order of derivation α. Consider, for instance, the triadic Cantor set C, built on the interval $[0, 1]$, whose dimension is $d = \ln 2/\ln 3$. Let $1_C(x)$ be the function whose value is one in the points belonging to the Cantor set upon $[0, 1]$, zero elsewhere. Therefore, the solution of $D^\alpha f(x) = 1_C(x)$ when $\alpha = d$ is $f(x) = {}_aD_b^{-\alpha} 1_C(x)$. Applying (12) with $x_0 = 0$ and $x_N = x$ and choosing x_i^* to be such that $1_C(x_i^*)$ is maximum in the interval $[x_i, x_{i+1}]$, one gets [17]

$$ f(x) = {}_0D_x^{-\alpha} 1_C(x) = \lim_{N\to\infty} \sum_{i=0}^{N-1} F_C^i \frac{(x_{i+1}-x_i)^\alpha}{\Gamma(1+\alpha)} = \frac{S(x)}{\Gamma(1+\alpha)}, \tag{13} $$

where F_C^i is a flag function that takes value 1 if the interval $[x_i, x_{i+1}]$ contains a point of the set C and 0 otherwise; hence $S(x)$ is the Cantor (devil's) staircase (Fig. 1b). Moreover, eq. (13) introduces the fractional measure of a fractal set we were looking for: for the Cantor set C it is defined as $\mathcal{F}^\alpha(C) = {}_0D_1^{-\alpha} 1_C(x)$. In fact $\mathcal{F}^\alpha(C)$ is infinite if $\alpha < d$, and 0 if $\alpha > d$. For $\alpha = d$, we find $\mathcal{F}^\alpha(C) = 1/\Gamma(1+\alpha)$. This measure definition yields the same value of the dimension predicted by the Hausdorff one, the difference being represented only by a different value of the normalization constant.

Eventually, consider two continuous functions $f(x)$ and $g(x)$ defined upon $[a, b]$ with a zero first derivative except at the points belonging to the same lacunar fractal set C where they present an Hölder exponent α equal to the dimension of the fractal support (i.e., $f(x)$ and $g(x)$ are Cantor staircase type functions). Based on eq. (11), it can be proved that, in the singular points $x \in C$, (i) the product function $h(x) = f(x)g(x)$ has the same Hölder exponent α unless both the factor functions have zero value; (ii) the LFD of order α of $h(x)$ can be computed using the classical rule for the differentiation of the product:

$$ D^\alpha h(x) = f(x) D^\alpha g(x) + g(x) D^\alpha f(x). \tag{14} $$

Performing now, for both the sides of eq. (14), a fractal integration of order α upon $[a, b]$ yields to the following *fractal integration by parts*:

$$ {}_aD_b^{-\alpha}[f(x) D^\alpha g(x)] = [h(b) - h(a)] - {}_aD_b^{-\alpha}[g(x) D^\alpha f(x)] \tag{15} $$

which will be useful in the next section.

4 Kinematic and static equations for fractal media

As shown in the Sects. 1 and 2, fractality plays a very important role in the mechanics of materials with an heterogeneous microstructure. The aim of this Section is to develop a model that, by the local fractional operators introduced in Sect. 3, is able to capture intrinsically the fractality of the material and, consequently, the size effects upon the related physical quantities. Thus, let us start with a uniaxial model [26], hereafter called *fractal Cantor bar* according to Feder's terminology [19]. Hence, consider a specimen of disordered material of length b. Suppose now to apply a tensile load in the z (axial) direction. As pointed out in Sect. 2, because of the fractal localization of strain, the plot of the axial displacement w versus z is a Cantor staircase (Fig. 1b). This plot corresponds to a strain field which is zero almost everywhere (corresponding to the integer portions) except in an infinite number of points where it is singular (corresponding to the localized cracks). The displacement singularities can be characterized by the LFD of order equal to the fractal dimension $\alpha = 1 - d_\varepsilon$ of the domain of the singularities, the unique value for which the LFD is finite and different from zero (the critical value). This computation is equivalent to eq. (3), passing from the global level to the local one. Therefore, we can define analytically the fractal strain ε^* as the LFD of the displacement

$$ \varepsilon^*(z) = D^\alpha w(z). \tag{16} $$

Let us observe that, in eq. (16), the non-integer physical dimensions $[L]^{d_\varepsilon}$ of ε^* are introduced by the LFD, whilst in eq. (3) they are a geometrical consequence of the fractal dimension of the localization domain.

Now let us turn our attention to the differential equilibrium equation, when the fractal bar is subjected to an axial load. Consider again a fiber of the specimen and suppose that the body is in equilibrium, $z = 0$ and $z = b$ being its extreme cross sections. We indicate with $p^*(z)$ the axial load per unit of fractal length acting upon the fractal bar and with $N(z)$ the axial force acting on the generic cross section orthogonal to the z-axis. Take therefore into consideration a kinematical field (w, ε^*) satisfying eq. (16) and a static field (N, p^*). The fractal integration by parts (15) can be interpreted as the principle of virtual work for the fractal bar. In fact, according to the fractal nature of the material microstructure, the internal virtual work can be computed as the fractal α-integral of the product of the axial force N times the fractal strain ε^* performed over the interval $[0, b]$, which, according to eqs. (16) and (15), is in its turn equal to

$$ {}_0D_b^{-\alpha}[N(z)\varepsilon^*(z)] = {}_0D_b^{-\alpha}[N(z)D^\alpha w(z)] = [N(z)w(z)]_{z=0}^{z=b} - {}_0D_b^{-\alpha}[w(z)D^\alpha N(z)]. \tag{17} $$

Since the body is in equilibrium, the virtual work principle holds. Hence the right hand side of eq. (17) must be equal to the external virtual work. This is true if and only if

$$ D^\alpha N(z) + p^*(z) = 0 \tag{18} $$

which is the (fractional) static axial equation of the fractal bar. Observe the anomalous dimension of the load p^*, $[F][L]^{-(1-d_\varepsilon)}$, since it considers forces acting on a fractal medium.

What has been done in the one-dimensional case can be formally extended in the three-dimensional case for a generic fractal medium [27]. As in the classical continuum mechanics, one needs the introduction of the fractal stress $\{\sigma^*\}$ and fractal strain $\{\varepsilon^*\}$ vectors to replace the corresponding scalar quantities in eqs. (16) and (18). Denoting with $\{\eta\}$ the displacement vector, the kinematic equations for a fractal medium can be expressed as

$$ \{\varepsilon^*\} = [\partial^\alpha]\{\eta\}, \tag{19} $$

where $[\partial^\alpha]$ is the kinematic fractional differential operator containing local fractional derivatives of order $\alpha = 1 - d_\varepsilon$. Eq. (19) is the three-dimensional extension of eq. (16). Analogously, eq. (18) becomes

$$ [\partial^\alpha]^T \{\sigma^*\} = -\{\mathcal{F}^*\}, \tag{20} $$

where $[\partial^\alpha]^T$ is the static fractional differential operator, transposed of the kinematic one and $\{\mathcal{F}^*\}$ is the vector of the forces per unit of fractal volume. From the physical dimension of the matrices at the first hand side of eq. (20) and from the fundamental relationship (6) among the fractal exponents, it can be easily shown that $\{\mathcal{F}^*\}$ owns the following physical dimension: $[F][L]^{-(2+d_\mathcal{G})}$, where $(2 + d_\mathcal{G})$, comprised between 2 and 3, should now be seen as the fractal dimension of the fractal medium.

In order to get the expression of the principle of the virtual work for a fractal medium, we need the extension to fractal domain of the Green theorem. This extension can be obtained performing a fractal integration of order $\beta - \alpha$ of both sides of eq. (15):

$$ D_{\Omega^*}^{-\beta}[fD^\alpha g] = D_{\Gamma^*}^{-(\beta-\alpha)}[fgn_x] - D_{\Omega^*}^{-\beta}[gD^\alpha f], \tag{21} $$

where now D^α is the LFD in the x-direction, n_x is the x-component of the outward normal vector to the fractal boundary Γ^* of the fractal body Ω^*. Other two scalar expressions can be obtained analogously to eq. (21), just considering the LFDs in the y and z-directions. Thus we are now able to derive the expression of the principle of virtual work for fractal media. It is sufficient to apply the extension of the Green theorem – eq. (21) – substituting appropriately to the functions f, g the components of the fractal stress $\{\sigma^*\}$ and displacement $\{\eta\}$ vectors. Furthermore, α and β are equal respectively to $(1 - d_\varepsilon)$ and $(2 + d_\mathcal{G})$. Thus for vector fields $\{\sigma^*\}$, $\{\mathcal{F}^*\}$ satisfying eq. (20) (i.e. statically admissible) and vectors fields $\{\varepsilon^*\}$, $\{\eta\}$ satisfying eq. (19) (i.e. kinematically admissible), it is possible to prove the validity of the following equation:

$$ \int_{\Omega^*} \{\mathcal{F}^*\}^T \{\eta\} \mathrm{d}\Omega^* + \int_{\Gamma^*} \{p^*\}^T \{\eta\} \mathrm{d}\Gamma^* = \int_{\Omega^*} \{\sigma^*\}^T \{\varepsilon^*\} \mathrm{d}\Omega^* \tag{22} $$

which represents the principle of virtual work for a generic fractal medium and is the natural extension of the classical continuum mechanics formulation of the principle. For the sake of clarity, in eq. (22) we used the classical symbol for the integrals; anyway they are fractal integrals over fractal domains. $\{p^*\}$ is the vector of the contact forces acting upon the (fractal) boundary of the fractal medium; it has the same physical dimension of the fractal stress, to which it is related by the relation

$$ [\mathcal{N}]^T \{\sigma^*\} = \{p^*\} \tag{23} $$

as naturally comes out in the proof of eq. (22). $[\mathcal{N}]^T$ is defined at any dense point of the boundary as the cosine matrix of the outward normal vector to the boundary of the initiator (see [19]) of the fractal set occupied by the body.

5 Conclusions

In this paper, the topologic framework for the mechanics of deformable fractal media has been outlined. Based on the experimental observations of the size effects on the parameters characterizing the cohesive law of materials with a disordered microstructure, the fractal quantities characterizing the process of deformation have been pointed out. In the second part of the paper, new mathematical operators from fractional calculus have been applied to write the field equations for solids with a fractal microstructure. It has been shown that the classical fractional calculus cannot be used to describe properly the deformations of fractal media. Instead, the local fractional operators, recently introduced by Kolwankar [8], can be successfully applied for our purposes. The static and kinematic equations for fractal media have been obtained. Moreover, the extension of the Green Theorem to fractal quantities and domains has been proposed, naturally yielding the Principle of Virtual Work for fractal media. The next step should be the definition of proper constitutive laws (e.g. elasticity) for fractal media. At this stage, only the formal structure of the static and kinematic equations has been outlined. Moreover, further analytical research about local fractional operators has to be carried out. Thus, engineering calculations may only be at an early stage. However, once these goals were achieved, boundary value problems on fractal sets could be solved, not only in principle, by means of the Local Fractional Calculus.

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