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A fractional calculus approach to the description of stress and strain localization in fractal media

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Abstract

Evidence of fractal patterns in materials with disordered microstructure under tensile loads is undeniable. Unfortunately fractal functions cannot be solution of classical differential equations. Hence a new calculus must be developed to handle fractal processes. In this paper, we use the local fractional calculus operators recently introduced by K.M. Kolwankar [Studies of fractal structures and processes using methods of fractional calculus. PhD thesis, University of Pune, India, 1998]. Through these new mathematical tools we get the static and kinematic equations that model the uniaxial tensile behavior of heterogeneous materials. The fractional operators respect the non-integer (fractal) physical dimensions of the quantities involved in the governing equations, while the virtual work principle highlights the static-kinematic duality among them. The solutions obtained from the model are fractal and yield to scaling power laws characteristic of the nominal quantities, i.e., they reproduce the size effects due to stress and strain localization.

1. Introduction

After the publication of the book by Mandelbrot [2], fractals have found several applications in Science as well as in Engineering. We can quote works in various fields ranging from growth phenomena to turbulence, from chaotic dynamical systems to image analysis. The main characteristic of fractals is their irregularity over all the length scales. This irregularity is the reason of the non-integer dimensions of fractal sets and, unfortunately, it makes them very difficult to handle analytically since the usual calculus is inadequate to describe such structures and processes. Fractals are too irregular to have any smooth differentiable function defined on them. Fractal functions do not possess first-order derivative at any point. Therefore it is argued that a new calculus should be developed which includes intrinsically a fractal structure [3]. Recently, Kolwankar [1], based on fractional calculus, defined new mathematical operators that appear to be useful in the description of fractal processes. It is important to emphasize that, what seems to be really interesting in studying fractals via fractional calculus, are the non-integer physical dimensions that arise dealing with both fractional operators and fractal sets. Physically, this means to find the same scaling laws both from an analytic and a geometric point of view.

The aim of the present paper is to show a possible, simple application in solid mechanics of the new operators introduced by Kolwankar [1]. We will present a model for the description of the static and kinematic fields of a material that shows fractal patterns under tensile stresses, as often happens for heterogeneous microstructured materials. The model takes into account this fractality intrinsically through the

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fractional calculus operators that appear in the governing equations. In the authors' opinion, this model should be a preliminary application in view of more advanced solid mechanics problems whose fractal features have been experimentally detected. Among these, we cite the slip band formation in metals [4], the fracture patterns in concrete and rocks [5], the stick-slip process in the frictional contact of rough interfaces [6], the shrinkage cracking net in dry soil.

The plan of the paper is as follows. In Section 2, we start by reviewing the conventional fractional calculus and its possible link with fractal geometry. Then we devote Section 3 to the local fractional calculus, anouncing some of its properties that will be useful in the following sections. Section 4 states the governing equations of the model. In Section 5, the model is solved for the case of constant normal force, highlighting analytically the scaling laws the main quantities are subjected to.

2. Conventional fractional calculus

Classical calculus treats integrals and derivatives of integer order. Fractional calculus is the branch of mathematics that deals with the generalization of integrals and derivatives to all real (and even complex) orders. There are various definitions of fractional differintegral operators not necessarily equivalent to each other. A complete list of these definitions can be found in the fractional calculus treatises [7–10]. These definitions have different origin. The most frequently used definition of a fractional integral of order q ($q > 0$) is the Riemann–Liouville's definition, which is a straightforward generalization to non-integer values of Cauchy's formula for repeated integration

$$\frac{\mathrm{d}^{-q}f(x)}{[\mathrm{d}(x-a)]^{-q}} = \frac{1}{\Gamma(q)}\int_a^x \frac{f(y)}{(x-y)^{1-q}}\,\mathrm{d}y. \tag{1}$$

From this formula, it appears logical to define the fractional derivative of order $n-1 < q < n$ (n integer) as the nth integer derivative of the $(n-q)$th fractional integral

$$\frac{\mathrm{d}^{q}f(x)}{[\mathrm{d}(x-a)]^{q}} = \frac{1}{\Gamma(n-q)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x \frac{f(y)}{(x-y)^{q-n+1}}\,\mathrm{d}y. \tag{2}$$

Once these definitions are given, it is natural to write differential equations in terms of such quantities. In the last decade, many fractional differential equations have been proposed. They include relaxation equations, wave equations, diffusion equations, etc. [11]. In these generalizations, one replaces the usual integer order time derivatives by fractional ones. In such way, by varying the order of derivation, it is possible to obtain a continuous transition between completely different models of the mathematical physics.

Of course, when q is not a positive integer, the fractional derivative (2) is a non-local operator since it depends on the lower integration limit a. The chain rule, Leibniz rule, composition law and other properties have been studied for the fractional derivatives [7]. Looking for a link between fractional calculus and fractals, it is worthwhile to cite the following scaling property (for $a = 0$)

$$\frac{\mathrm{d}^{q}f(bx)}{[\mathrm{d}x]^{q}} = b^q\frac{\mathrm{d}^{q}f(bx)}{[\mathrm{d}(bx)]^{q}}. \tag{3}$$

It means that the fractional differintegral operators are subjected to the same scaling power laws the quantities defined on fractal domains are subjected to (q being the fractal dimension). For the scaling property in the case $a \neq 0$, see [7].

More recently, another important result has been achieved concerning the maximum order of fractional differentiability for non-classical differentiable (fractal) functions. Let us explain this property for two kinds of functions: the Weierstrass function and the Cantor staircase (sometimes called devil's staircase). The first function is continuous but nowhere differentiable. The singularities are locally characterized by the Hölder exponent, which is everywhere constant and equal to a certain value $0 < s < 1$. It is possible to prove that the graph of this function is fractal with a box-counting dimension $2-s$ and hence greater than 1. Although fractal, the Weierstrass function admits continuous fractional derivatives of order lower than s.

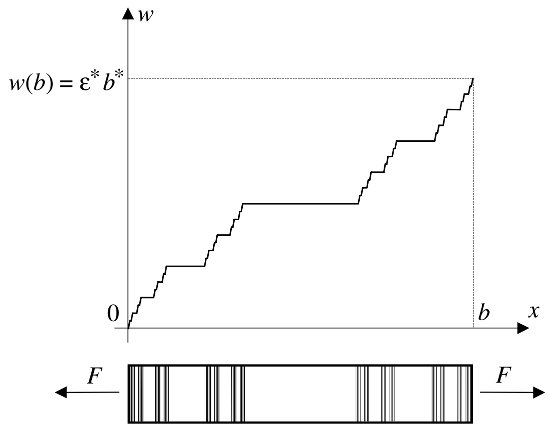

Fig. 1. Fractal strain localization and related Cantor staircase displacement graph.

Hence there is a direct relationship between the fractal dimension of the graph and the maximum order of differentiability: the greater the fractal dimension, the lower the differentiability.

A devil's staircase [12] type function (Fig. 1) can be obtained as the integral of a constant mass density upon a lacunar fractal set belonging to the interval [0, 1]. It is a monotonic function that grows on a fractal support; elsewhere it is constant. This kind of functions are not fractal since they present a finite length; on the other hand, they have an infinite number of singular points characterized by a Hölder exponent equal to the fractal dimension of the support. Schellnhuber & Seyler [13] proved that the Cantor staircase admits continuous fractional derivatives of order lower than the fractal dimension of the set where it grows.

From a physical point of view, some efforts have been done to apply space fractional differential equations to the study of phenomena involving fractal distributions in space. Here we can quote Giona & Roman [14], who proposed a fractional equation to describe diffusion on fractals, and Nonnenmacher [15], who showed that a class of Lévy type processes satisfies an integral equation of fractional order. This order is also the fractal dimension of the set visited by a random walker whose jump size distribution follows the given Lévy distribution.

3. Local fractional calculus

Recently, a new notion called *local fractional derivative* (LFD) has been introduced with the motivation of studying the local properties of fractal structures and processes [16]. The LFD definition is obtained from (2) introducing two "corrections" in order to avoid some physically undesirable features of the classical definition. In fact, if one wishes to analyze the local behavior of a function, both the dependence on the lower limit a and the fact that adding a constant to a function yields to a different fractional derivative should be avoided. This can be obtained by subtracting from the function the value of the function at the point where we want to study the local scaling property and choosing the lower limit as that point itself. Therefore, restricting our discussion to an order q comprised between 0 and 1, the LFD is defined as the following limit (if it exists and is finite):

$$\mathrm{D}^q f(y) = \lim_{x \to y} \frac{\mathrm{d}^q (f(x) - f(y))}{[\mathrm{d}(x-y)]^q}, \quad 0 < q \leqslant 1. \tag{4}$$

In [16] it has been shown that the Weierstrass function is locally fractionally differentiable up to a "critical order" α between 0 and 1. More precisely, the LFD is zero if the order is lower than α, does not exist if greater, while exists and is finite only if equal to α. Thus, the LFD shows a behavior analogous to the Hausdorff measure of a fractal set. Furthermore, the critical order is strictly linked to the fractal properties of the function itself. In fact, Kolwankar & Gangal [16] showed that the critical order is equivalent to the local Hölder exponent (which depends, as we have seen, on the fractal dimension), by proving the following local fractional Taylor expansion of the function $f(x)$ of order q (with $N < q < N+1$, N natural, q real) for $x \to y$ [17,18]:

$$f(x) = \sum_{n=0}^{n=N} \frac{f^{(n)}(y)}{\Gamma(n+1)}(x-y)^n + \frac{\mathrm{D}^q f(y)}{\Gamma(q+1)}(x-y)^q + R_q(x-y), \tag{5}$$

where R_q is a remainder, negligible if compared with the other terms. If $q < 1$, the previous expression becomes

$$f(x) = f(y) + \frac{\mathrm{D}^q f(y)}{\Gamma(q+1)}(x-y)^q + R_q(x-y). \tag{6}$$

Let us observe that the terms in the right-hand side of Eqs. (5), (6) are non-trivial and finite only if q is equal to the critical order α. Moreover, for $q = \alpha$, the fractional Taylor expansion (6) gives us the geometrical interpretation of the LFD. When q is set equal to unity, one obtains from (6) the equation of a tangent. All the curves passing through the same point y with the same first derivative have the same tangent. Analogously, all the curves with the same critical order α and the same D^α form an equivalence class modeled by x^α. This is how it is possible to generalize the geometric interpretation of derivatives in terms of "tangents".

Consider now two functions $f(x)$ and $g(x)$ having the same Hölder exponent α at the same point $x = y$. Based on Eq. (6), it is easy to prove that: (i) in $x = y$ the product function $h(x)$ has the same Hölder exponent α unless both the factor functions have zero value; (ii) in $x = y$ the LFD of order α of the product function can be computed using the classical rule for the differentiation of the product. That is:

$$\mathrm{D}^\alpha h(y) = f(y)\mathrm{D}^\alpha g(y) + g(y)\mathrm{D}^\alpha f(y). \tag{7}$$

The solution of the simple differential equation $\mathrm{d}f/\mathrm{d}x = 1_{[0,x]}$ gives the length of the interval $[0,x]$. The solution is nothing but the integral of the unit function. Wishing to extend this idea to the computation of the measure of fractal sets, it can be seen immediately that the fractional integral (1) does not work as it fails to be additive because of its non-trivial kernel. On the other hand, Kolwankar [19] proved that a fractal measure can be obtained through the inverse of the LFD defined as:

$${}_a\mathrm{D}_b^{-\alpha} f(x) \equiv \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i^*) \frac{\mathrm{d}^{-\alpha} 1_{\mathrm{d}x_i}(x)}{[\mathrm{d}(x_{i+1}-x_i)]^{-\alpha}}, \tag{8}$$

where $[x_i, x_{i+1}],\ i = 0, \ldots, N-1,\ x_0 = a$ and $x_N = b$, provide a partition of the interval $[a,b]$ and x_i^* is some suitable point chosen in the subinterval $[x_i, x_{i+1}]$, while $1_{\mathrm{d}x_i}$ is the unit function defined on the same subinterval. Kolwankar called ${}_a\mathrm{D}_b^{-\alpha} f(x)$ the *fractal integral* of $f(x)$ over the interval $[a,b]$.

The simple local fractional differential equation $\mathrm{D}^\alpha f(x) = g(x)$ has not a finite solution when $g(x)$ is constant and $0 < \alpha < 1$. Interestingly, the solution exists if $g(x)$ has a fractal support whose Hausdorff dimension d is equal to the fractional order of derivation α. Consider, for instance, the triadic Cantor set C, built on the interval $[0,1]$, whose dimension is $d = \ln 2/\ln 3$. Let $1_C(x)$ be the function whose value is one in the points belonging to the Cantor set upon $[0,1]$, zero elsewhere. Therefore, the solution of $\mathrm{D}^\alpha f(x) = 1_C(x)$ when $\alpha = d$ is $f(x) = {}_0\mathrm{D}_x^{-\alpha} 1_C(x)$. Applying (8) with $x_0 = 0$ and $x_N = x$ and choosing x_i^* to be such that $1_C(x_i^*)$ is maximum in the interval $[x_i, x_{i+1}]$, one gets [17]

$$f(x) = {}_0\mathrm{D}_x^{-\alpha} 1_C(x) = \lim_{N\to\infty} \sum_{i=0}^{N-1} F_C^i \frac{(x_{i+1}-x_i)^\alpha}{\Gamma(1+\alpha)} = \frac{S(x)}{\Gamma(1+\alpha)}, \tag{9}$$

where F_C^i is a flag function that takes value 1 if the interval $[x_i, x_{i+1}]$ contains a point of the set C and 0 otherwise; hence $S(x)$ is the Cantor (devil's) staircase (Fig. 1). Moreover, Eq. (9) introduces a fractional measure of the Cantor set defined as $\mathscr{F}^\alpha(C) = {}_0\mathrm{D}_1^{-\alpha} 1_C(x)$. In fact $\mathscr{F}^\alpha(C)$ is infinite if $\alpha < d$, and 0 if $\alpha > d$. For $\alpha = d$, we find $\mathscr{F}^\alpha(C) = 1/\Gamma(1+\alpha)$. This measure definition yields the same value of the dimension predicted by the Hausdorff one, the difference being represented only by a different value of the normalization constant.

Eventually, consider two continuous functions $f(x)$ and $g(x)$ defined upon $[a,b]$ with a zero first derivative except at the points belonging to the same fractal set where they present an Hölder exponent α equal to the dimension of the fractal support (i.e., $f(x)$ and $g(x)$ are Cantor staircase type functions). Hence Eq. (7) is valid on the whole interval $[a,b]$. Performing, for both the sides of Eq. (7), a fractal integration of order α upon $[a,b]$ yields to the following *fractal integration by parts*:

$$ {}_a\mathrm{D}_b^{-\alpha}[f(x)\mathrm{D}^\alpha g(x)] = [h(b) - h(a)] - {}_a\mathrm{D}_b^{-\alpha}[g(x)\mathrm{D}^\alpha f(x)], \tag{10} $$

which will be utilized in the next section.

4. The problem of the fractal bar

The fractal nature of damage and porosity in concrete [20] and rocks [21] has been experimentally detected over a wide range of scales. It seems therefore appropriate to describe disordered media with fractal models. The archetype of fractal media is the Menger sponge (Fig. 2), a well-known deterministic fractal whose dimension is lower than three. Obviously, real media such as concrete are stochastic fractals, which, as the Menger sponge, show pores and voids at all the scales of observation. Henceforth we will indicate with $2+\alpha$ the non-integer dimension of the fractal medium. Consequently, $1+\alpha$ will be the fractal dimension of the generic cross-section and α the dimension of the generic fiber.

The aim of this section is to develop a model that, by the local fractional operators introduced in Section 3, is able to capture the fractality of the material and, consequently, the size effects upon the related physical quantities. For the sake of simplicity, we will restrict ourselves to a uniaxial model, hereafter called *fractal Cantor bar* according to Feder's terminology [12].

Hence, consider a specimen of disordered material of size b. Suppose now to apply a tensile stress σ in the x direction. The extreme of the generic fiber along x will undergo a displacement u equal to the strain ε times the length b. On the other hand, refining the scale of observation, it can be noticed that, due to diffused fractal cracking, the deformation concentrates into smaller and smaller regions (Fig. 1). The total elongation must be resolution-independent. The product $\varepsilon_i b_i$ must therefore be constant at each scale i of observation and, in the limit of the finest resolution ($i \to \infty$), $\varepsilon_i \to \infty$ and $b_i \to 0$. The renormalization group (RG) procedure can be applied to the nominal strain, yielding

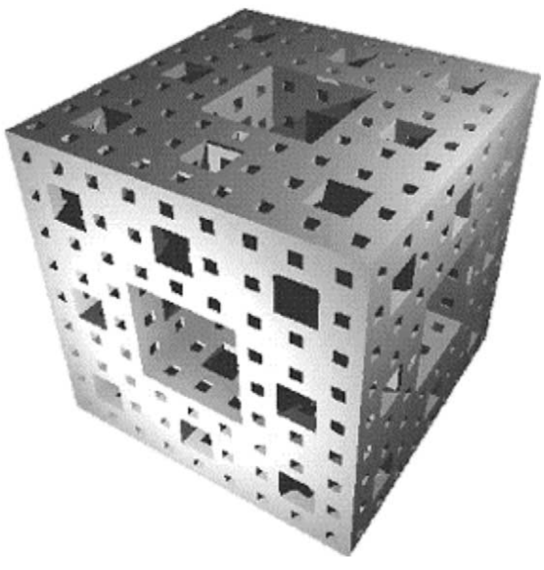

Fig. 2. The Menger sponge is the archetype of fractal porous media; its fractal dimension is $d \cong 2.727$.

$$u = \varepsilon b = \varepsilon_1 b_1 = \varepsilon_2 b_2 = \cdots = \varepsilon_i b_i = \cdots = \varepsilon^* b^*. \tag{11}$$

According to the RG procedure, in order to overcome the presence of diverging and vanishing terms and to keep dimensional homogeneity, the micro-scale description requires the product of an anomalous (fractal) strain ε^* times the fractal length b^* of the fiber. b^* has dimension $[\mathrm{L}]^\alpha$, while its measure, according to the definition of fractional measure in Section 3, is $b^\alpha/\Gamma(1+\alpha)$. Differently from the classical strain ε, which is only a nominal quantity, the fractal strain ε^* is scale-independent and its physical dimension is $[\mathrm{L}]^{1-\alpha}$.

Because of the fractal localization of strain, the plot of the displacement u versus x is a Cantor staircase (Fig. 1). This plot corresponds to a strain field which is zero almost everywhere (corresponding to the integer portions) except in an infinite number of points where it is singular (corresponding to the localized cracks). The displacement singularities can be characterized by the LFD of order equal to the fractal dimension α of the support, the unique value for which LFD is finite and different from zero (the critical value). This computation is equivalent to the RG procedure (11), passing from the global level to the local one. Therefore, we can define the fractal strain ε^* also as the LFD of the displacement

$$\varepsilon^*(x) = \mathrm{D}^\alpha u(x). \tag{12}$$

Let us observe that, in Eq. (12), the non-integer physical dimensions of ε^* are introduced through the fractional operator, whilst in Eq. (11) they are a geometrical consequence of the fractal dimension of the support.

Now we turn our attention to the differential equilibrium equation, when the fractal bar is subjected to an axial load. Consider again a fiber of the specimen and suppose that the body is in equilibrium, $x = 0$ and $x = b$ being its extreme cross-sections. We indicate with $p^*(x)$ the axial load per unit of fractal length acting on the bar and with $N(x)$ the axial force acting on the generic cross-section orthogonal to the x-axis. Take therefore into consideration a displacement field (u, ε^*) satisfying Eq. (12) and a stress field (N, p^*). According to the fractal nature of the bar, the virtual internal work can be computed as the fractal α-integral of the product of the axial force N times the fractal strain ε^* performed over the interval $[0, b]$, which, according to Eqs. (12) and (10), is in its turn equal to

$$ {}_0\mathrm{D}_b^{-\alpha}[N(x)\varepsilon^*(x)] = {}_0\mathrm{D}_b^{-\alpha}[N(x)\mathrm{D}^\alpha u(x)] = [N(x)u(x)]_{x=0}^{x=b} - {}_0\mathrm{D}_b^{-\alpha}[u(x)\mathrm{D}^\alpha N(x)]. \tag{13}$$

Since the body is in equilibrium, the virtual work principle holds. Hence the right-hand side of Eq. (13) must be equal to the virtual external work. This is true if and only if

$$\mathrm{D}^\alpha N(x) + p^*(x) = 0, \tag{14}$$

which is the (fractal) static axial equation of the fractal bar. Observe the anomalous dimension of the load p^*, $[\mathrm{F}][\mathrm{L}]^{-\alpha}$, since it considers forces acting on a fractal medium.

Once the axial force N is known, the computation of the stress acting upon a generic cross-section undergoes the same indetermination encountered for the strain, because of the lacunar fractal nature of the cross-section itself. In Fig. 3 are reported a stochastic fractal representation of a concrete specimen cross-section and its deterministic counterpart (the Sierpinsky carpet). At first sight, one can say that the force is equal to the tensile stress σ times the cross-sectional area $A = b^2$. On the other hand, refining the scale of observation, it can be noticed that, due to multiscale voids and cracks, the stress appears to concentrate into regions smaller and smaller. The force N is resolution-independent; hence, the products $\sigma_i A_i$ at each scale i of observation, must be constant. In the limit of the finest resolution $(i \to \infty)$, $\sigma_i \to \infty$ and $A_i \to 0$. As in Eq. (11), the RG procedure can be applied to the nominal stress

$$N = \sigma A = \sigma_1 A_1 = \sigma_2 A_2 = \cdots = \sigma_i A_i = \cdots = \sigma^* A^*. \tag{15}$$

At the micro-scale level, the axial force is the product of an anomalous (fractal) stress σ^* times the fractal area A^* of the cross-section, which presents a dimension $[\mathrm{L}]^{1+\alpha}$ and hence a fractional measure

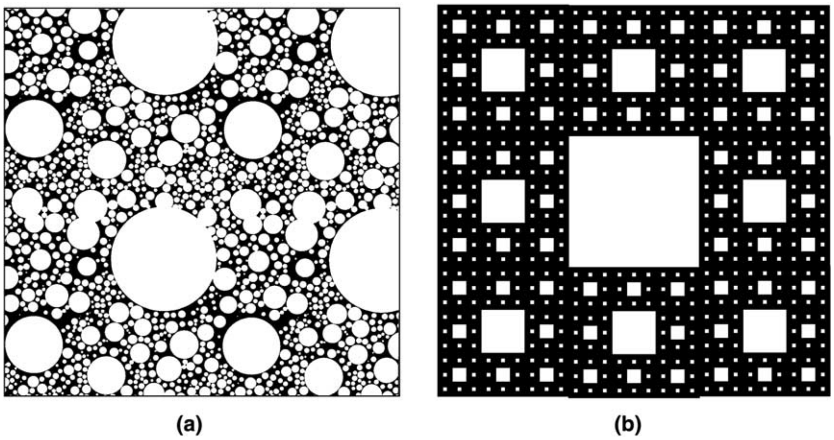

Fig. 3. Stochastic (a) and deterministic (b) representation of a concrete specimen cross-section.

$b^{1+\alpha}/\Gamma(1+\alpha)$. While the nominal stress σ depends on the scale of observation, the fractal stress σ^*, with physical dimensions $[\mathrm{F}][\mathrm{L}]^{-(1+\alpha)}$, is the true scale-invariant quantity.

5. The example of a concentrated force applied at the free extremity of a clamped fractal bar

In Section 4, we have shown how a scale-independent characterization of a stretched specimen can be achieved describing the static and kinematic fields in terms of fractal stress and fractal strain. In order to get some basic solutions, we will consider Hooke's law, i.e., $\sigma = E\varepsilon$. From the RG procedures (11) and (15), it is easy to observe that the modulus of elasticity E is also the coefficient of the linear relationship between the fractal quantities, i.e., $\sigma^* = E\varepsilon^*$. This relation allows us to highlight the size effects upon strength and deformability of the fractal bar.

Here we consider only the case in which the fractal bar is stretched in the x direction by a concentrated axial force N, assuming the body to be clamped in $x = 0$ and to be free at the end $x = b$, where the external force N is applied (Fig. 4). When the distributed external axial force p^* is zero, Eq. (14) tells us that the internal axial force is constant and equal to N throughout the bar. Through Eq. (15) and the

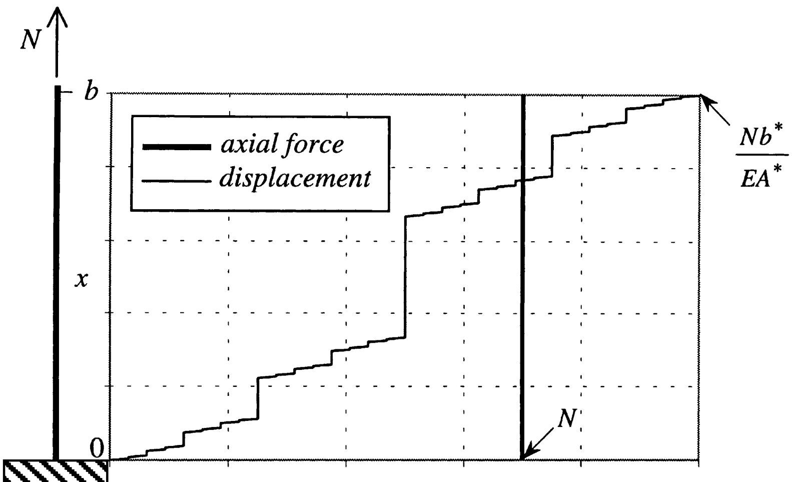

Fig. 4. The fractal bar subjected to a concentrated tensile load at its free end: axial displacement and axial force solutions.

constitutive link we get a fractal strain ε^* equal to N/EA^* where it exists. Hence the kinematic Eq. (12) becomes

$$\mathrm{D}^{\alpha}u(x) = \frac{N}{EA^*}1_C(x/b). \tag{16}$$

Introducing the dimensionless quantities $U = u/b,\ X = x/b\ (X \in [0,1])$, we can apply the scaling property (3), which is valid also for the LFD, to get $\mathrm{D}^{\alpha}u(x) = b^{1-\alpha}\mathrm{D}^{\alpha}U(X)$. Eq. (16) can therefore be expressed in dimensionless form as follows:

$$\mathrm{D}^{\alpha}U(X) = \frac{N}{EA^*b^{1-\alpha}}1_C(X). \tag{17}$$

The solution of Eq. (17) is obtained from Eq. (9) (the argument of the gamma function being constant, we will not indicate it explicitly, i.e., $\Gamma(1+\alpha) = \Gamma$)

$$U(X) = \frac{N}{EA^*b^{1-\alpha}}\frac{S(X)}{\Gamma}, \tag{18}$$

where $S(X)$ is the Cantor staircase. Recovering the physical quantities yield

$$u(x) = \frac{Nb^*}{EA^*}S(x/b). \tag{19}$$

The result is shown in Fig. 4. Let us observe that the Cantor staircase, introduced geometrically in Section 4, is now obtained analytically.

In order to get the scaling effects the fractal medium is subjected to, we fix the nominal stress $\sigma = N/A$ in the x direction and let b to vary; i.e., the structural size changes, the body shape and the stress being the same.

First, we analyze the size effect on tensile strength. From Eq. (15) it is easy to see that the fractal stress increases increasing the size: $\sigma^* = \sigma\Gamma b^{(1-\alpha)}$. Since σ^* is the scale-invariant quantity, it governs the rupture phenomenon; therefore the specimen will fail when the fractal stress attains its critical value σ_u^*. As a consequence, larger structures will fail at a lower nominal stress σ_u. Its decrease with the size is governed by the following scaling power law:

$$\sigma_{\mathrm{u}} = \left(\frac{\sigma_{\mathrm{u}}^*}{\Gamma}\right)b^{-(1-\alpha)}. \tag{20}$$

Eq. (20) was firstly obtained by Carpinteri [22] in a slightly different form and represents the proper explanation of the disordered materials structural weakening when the size increases.

The second size effect we are going to take into account concerns the deformability of the fractal medium at the failure point. Increasing the specimen size, failure occurs at a lower nominal strain. In fact, since the fractal stress at the critical load N_u (i.e., the fractal strength) must be the same for specimens of any size and equal to $\sigma_u^* = N_u/A^*$, from Eq. (19) the scaling law for the displacement $u_u(b)$ and the strain ε_u at rupture is straightforward [23]

$$\varepsilon_{\mathrm{u}} = \frac{u_{\mathrm{u}}(b)}{b} = \left(\frac{\sigma_{\mathrm{u}}^*}{E\Gamma}\right)b^{-(1-\alpha)}. \tag{21}$$

Thus at larger scales we find a more brittle behavior for what concerns the ultimate strain. Since the exponent of the reference dimension b is negative, the specimen fails at a lower strain increasing the size. This structural embrittlement is in agreement with experimental results on concrete specimens [23,24], where the failure is neither strain- (as in continuum mechanics) nor displacement- (as in fracture mechanics) controlled, but it is in an intermediate situation [25,26].

The size effects represented by the scaling laws (20) and (21) are summarized in Fig. 5. In terms of fractal quantities, the constitutive relation is scale-independent. Thus, considering two specimens with different characteristic dimensions b_1 and b_2 $(b_1 < b_2)$, the plots σ^* versus ε^* are the same for both of them. On the

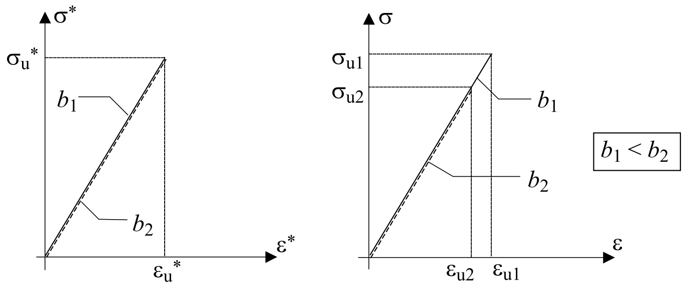

Fig. 5. Size effects on strength and deformability. Comparison between two specimens of characteristic dimensions b_1 and b_2 $(b_1 < b_2)$.

other hand, the plots are different if we consider nominal quantities. The larger specimen fails at a lower strain and at a lower stress.

6. Conclusions

Fractal functions cannot be solution of classical differential equations. It is argued that a new calculus should be developed to treat fractal processes and structures. The recently proposed local fractional calculus is the best candidate, whereas classical fractional calculus fails. The price to pay is high as local fractional operators have their own rules, which are different from those of Riemann–Liouville operators. The presented work should be a simple but consistent example of how it is possible to write equations governing processes taking place on fractal domains, in the respect of the non-integer physical dimensions of the quantities involved. The solutions of such equations yield to find the scaling power laws characterizing the fractal structures.

Acknowledgements

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