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# A scale-invariant cohesive crack model for quasi-brittle materials

Alberto Carpinteri *, Bernardino Chiaia, Pietro Cornetti

*Department of Structural and Geotechnical Engineering, Politecnico di Torino, Corso Duca Degli Abruzzi 24, 10129 Torino, Italy*

**Abstract**

The fictitious crack model by Hillerborg is the most widely used model to simulate damage and fracture in concrete structures. Its peculiar capability to capture the evolution of the cracking process is accompanied by its simplicity. However, some aspects of the phenomenon are not considered in the model, for instance the size-dependence of the nominal quantities involved in the cohesive law. This affects the predictive capabilities of the model, when it is used to extrapolate results from small laboratory specimens to full-scale structures.

In this paper, a scale-independent cohesive law is put forward, which overcomes these drawbacks and permits to obtain a unique constitutive relationship for softening in concrete. By assuming damage occurring in a fractal band inside the specimen, nominal stress, crack opening displacement and nominal fracture energy become scale dependent. Hence they should be substituted by fractal quantities, which are the true material constants. A mutual relation among their fractal physical dimensions puts a strong restriction to disorder. By varying the scaling exponents of the kinematical quantities, a clear transition from discrete to smeared cracking can be obtained. The fractal cohesive law is eventually applied to some tensile test data, showing perfect agreement between theory and experiments. 

*Keywords:* Concrete; Damage; Fractals; Fracture; Tension; Softening

## 1. Introduction

It is well known that strain and damage localization cause structural size effects in concrete, i.e. geometrically similar structures behave in different manners. The behavior of concrete, as well as of other materials with disordered microstructure, exhibiting damage localization (the so-called quasi-brittle materials), is described in a satisfactory way by the fictitious (or cohesive) crack model introduced by Hillerborg [1]. According to this model, the material can be characterized by a couple of constitutive laws: a stress–strain relationship ($\sigma$–$\varepsilon$), valid for the undamaged material, and a stress–crack opening displacement relationship ($\sigma$–$w$), the so-called cohesive law (Fig. 1). This law describes how the stress decreases from its maximum value to zero as the distance between the crack lips increases from zero to the critical displacement $w_c$. The area below the stress–crack opening displacement curve represents the energy spent to

* Corresponding author. Tel.: +39-011-564-4850; fax: +39-011-564-4899.
*E-mail address:* carpinteri@polito.it (A. Carpinteri).

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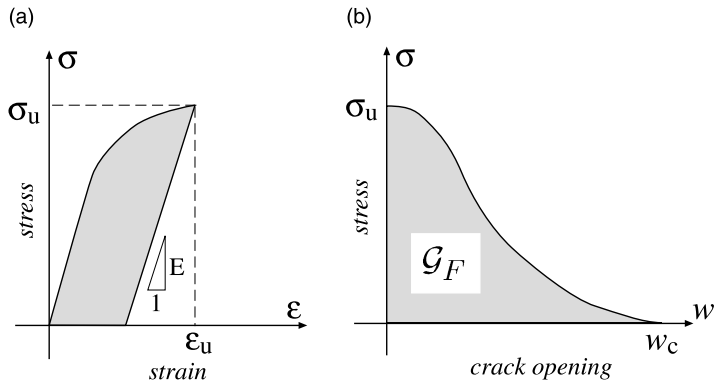


Fig. 1. Hillerborg's cohesive crack model: elastic stress–strain curve (a), and cohesive stress–crack opening displacement law (b).

create the unit crack surface ($\mathscr{G}_F$). If the two portions into which the specimen is separated undergo elastic unloading, the work done to split the specimen can be considered to be exactly equal to the product of $\mathscr{G}_F$ times the cracked area.

The cohesive crack model is able to explain different size effects encountered in concrete structures. More in detail, the model is able to simulate tests where high stress gradients are present, i.e. tests on pre-notched specimens. In these cases, the cohesive crack model captures the ductile–brittle transition occurring by increasing the size of the structure. On the other hand, relevant scale effects are encountered also in uniaxial tension tests on dog-bone shaped specimens, where much smaller stress gradients are present. In this case, size effects should be inherent to the material behavior rather than to the stress intensification.

Apart from the tests carried out by Bazant and Pfeiffer [2] in a limited scale range (1:4), uniaxial tensile tests on dog-bone shaped specimens were performed by Carpinteri and Ferro [3,4], with fixed boundary conditions, in a scale range 1:16, and by van Mier and van Vliet [5], with rotating boundary conditions, in a scale range 1:32. The tests, in both cases, proved that the physical parameters characterizing the cohesive law are scale dependent, thus showing the limits of Hillerborg's model. In fact, by increasing the size of the specimen, the ultimate stress decreases while the fracture energy increases.

Based on the experimental evidence, well-known models like the size effect law (SEL) by Bazant [6] and the multifractal scaling laws (MFSLs) by Carpinteri and co-authors [7–11] were developed in order to extrapolate results obtained at the laboratory scale to real concrete structures.

While Bazant's SEL (Fig. 2a) gives good predictions only for the strength of pre-notched specimens, the MFSLs (Fig. 2b) permit to calculate both the strength and fracture energy of large un-notched concrete structures. Recently, based on the hypothesis of fractal strain put forward by Cornetti [12], the MFSLs have been extended also to kinematical quantities (e.g. to the crack opening displacement).

The aim of the present paper is to provide a unified explanation of the scale-effects affecting the physical quantities of the cohesive law. If the strain localization occurs in a *fractal band*, the nominal quantities (ultimate strength, fracture energy, critical displacement) undergo size effects. More in detail, fractal strain localization allows us to explain the observed increasing tail of the cohesive law as the specimen size increases (see, for instance, Ref. [5]), i.e. it clarifies also the scaling of the critical displacement $w_c$. It is thus possible to introduce a fractal kinematical quantity, the *fractal critical strain* $\varepsilon_c^*$, which, together with the fractal maximum stress and the fractal fracture energy introduced by Carpinteri [13], represents one of the scale-independent quantities governing the damage process. The fractional exponents of these three quantities are strictly linked. By means of dimensional analysis, it is possible to demonstrate that only two of the three exponents are independent. In other words, there is a relationship between the exponents that provides a clear restriction to disorder.

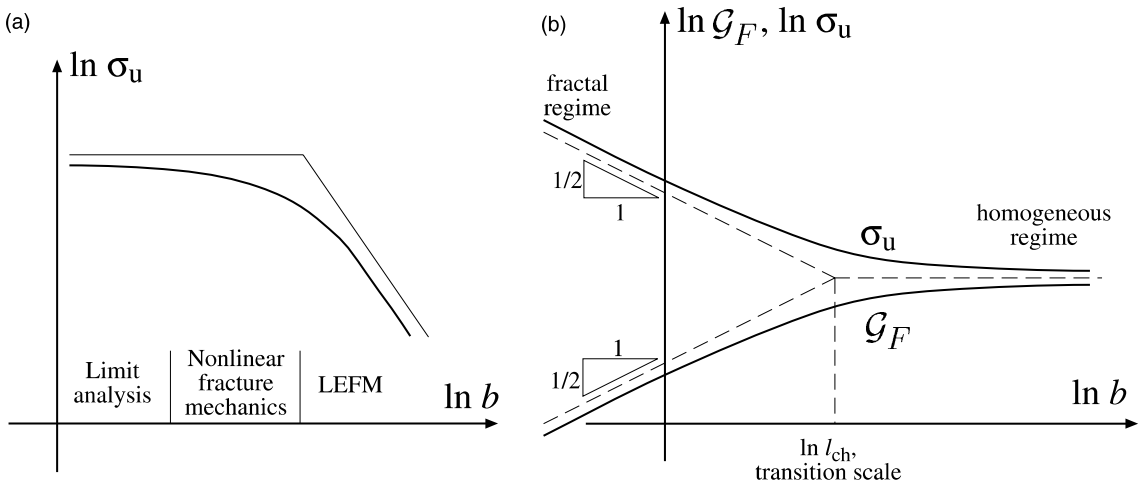


Fig. 2. Bazant's size effect law (a) compared to the multifractal scaling laws for nominal strength and fracture energy (b).

In terms of the new fractal quantities, it is possible to define a scale-invariant cohesive law, which represents the true material property. We thus call this model the *fractal* (*scale-independent*) cohesive crack model. The developed model is finally applied to the results of the tests carried out by Carpinteri and Ferro [3], in order to fit the fractal cohesive law to experimental data.

## 2. Fractal tensile strength

In continuum mechanics, we are concerned in the way through which forces are transmitted in a medium. The Cauchy definition of stress ($\sigma$) relies on some regularity properties (continuity and measurability) of the medium. Euclidean measurability implies the scale-independence of the mechanical laws. The classical elasticity theory is based on this assumption. On the other hand, when singularities and inhomogeneities are present, or when the structure of the body plays a fundamental role in the definition of the physical properties, a mesoscopic approach is required. This is the case of damage occurring in materials with heterogeneous microstructure (e.g. concrete and rocks), which present defects that interact, in a complex manner, at all scales.

Recent experimental results about the concrete porous microstructure [14] led us to believe that a consistent modelization of damage phenomena in concrete can be pursued by means of fractal geometry. In particular, *lacunar* fractal sets (possessing a fractal dimension $\Delta_\sigma$ lower than the topologic one) can be used to model phenomena occurring in the bulk, such as the stress flux through porous media subjected to tensile loading. From this point of view, the rarefied resisting cross-sections, in correspondence of the critical load, can be modeled by a stochastic lacunar fractal set of dimension $\Delta_\sigma$, with $\Delta_\sigma \leqslant 2$. Physically, this lacunarity is due to the self-similar distribution of voids, cracks and pores. Mathematically, the resisting cross-section can be characterized by the dimensional decrement $d_\sigma$ with respect to the Euclidean dimension $D = 2$. This implies: $\Delta_\sigma = 2 - d_\sigma$. From Fractal Geometry, we know that the apparent Euclidean measure (the area) of lacunar sets is scale dependent and tends to zero as the resolution increases. Finite measures can be obtained only with non-integer dimensions. These sets present zero area and infinite length. Thus, the Cauchy definition of stress cannot be applied. On the other hand, an original definition of fractal stress acting upon lacunar domains was put forward by Carpinteri [13].

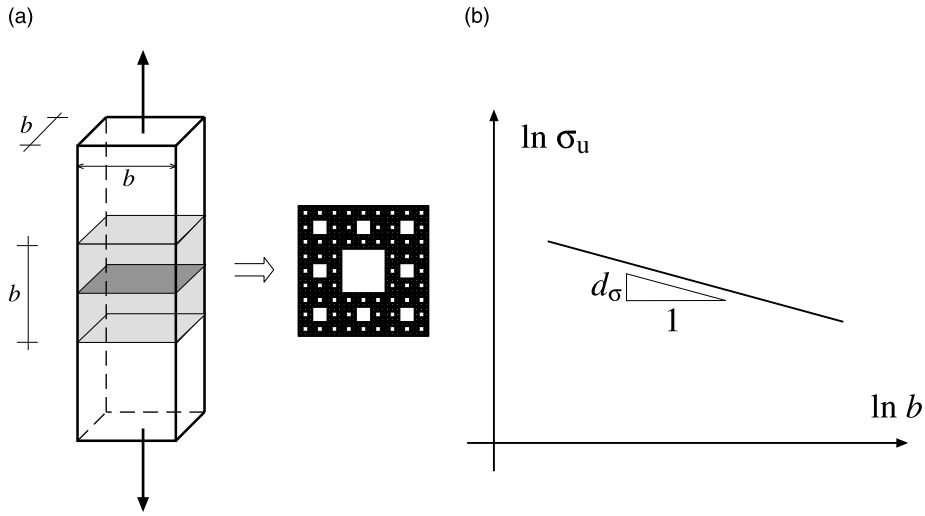


Fig. 3. Lacunarity of the cross-section (a) and scaling law for tensile strength $\sigma_u$ (b).

For the sake of simplicity, let us model the resisting cross-section with a deterministic fractal set, such as the Sierpinski carpet built on the $b$-sided square, as shown in Fig. 3a. The fractal dimension of this domain is $\Delta_\sigma = \ln 8/\ln 3 = 1.893$ ($d_\sigma = 0.107$). The canonical maximum stress $\sigma_u$ is computed as the ratio of the maximum load to the nominal area ($A_0 = b^2$) of the square. On the other hand, the effective area has not a unique value and is scale dependent. According to the fractal measure of the cross-section, the maximum force is given by the product of the Hausdorff measure $A^* \approx b^{(2-d_\sigma)}$ of the Sierpinski carpet times the *fractal tensile strength* $\sigma_u^*$:

$$F = \sigma_u A_0 = \sigma_u^* A^* \tag{1}$$

Details of this renormalization procedure can be found in Refs. [8,13]. Due to dimensional homogeneity, if $[L]^{2-d_\sigma}$ is the physical dimension of the fractal cross-section, then $\sigma_u^*$ possesses the anomalous physical dimensions $[F][L]^{-(2-d_\sigma)}$. The fractal tensile strength $\sigma_u^*$ is the true material constant, i.e. it is scale-invariant. From Eq. (1) we find that the maximum sustainable load increases with the size of the specimen according to $b^{(2-d_\sigma)}$ instead of $b^2$. Even more interesting, the fractal scaling law of the nominal tensile strength $\sigma_u$ can be extracted from Eq. (1) and put into logarithmic form:

$$\ln \sigma_u = \ln \sigma_u^* - d_\sigma \ln b \tag{2}$$

which implies linear scaling in the bilogarithmic diagram, with slope equal to $-d_\sigma$ (Fig. 3b). Eq. (2) represents the negative size effect on tensile strength, experimentally analyzed by Carpinteri et al. [9,10]. Experimental and theoretical results allow us to affirm that $d_\sigma$ can vary between the lower limit 0 (i.e. $\sigma_u^*$ holding its canonical dimensions $[F][L]^{-2}$ and absence of size effect on tensile strength), and the upper limit $1/2$ ($\sigma_u^*$ with the dimensions of a stress-intensity factor $[F][L]^{-3/2}$ and maximum size effect, which is the case of linear elastic fracture mechanics).

## 3. Fractal critical strain

Turning now our attention from a single cross-section to the whole specimen, it can be noticed that damage is often not localized onto a single section but is spread over a finite band. Based on this as-

sumption, Bazant and Oh [15] developed the crack band theory. They supposed that the thickness of the band where damage is localized is related to the maximum dimension of the aggregate, i.e. it is scale independent. Differently from Bazant's model, we do not make assumption on the band width, but simply assume that damage into heterogeneous media presents fractal patterns. This hypothesis, although rather anomalous, is not a simple abstraction in material science. Kleiser and Bocek [16] showed that in the bulk of some metals, when subjected to tension, the so-called slip-lines develop with typical fractal patterns. Poliakov et al. [17] measured fractal slip bands in highly stressed rock masses. Every damage band, when observed at a sufficiently high resolution, is made out of several smaller bands, which, in turn, appear to be constituted by smaller and smaller bands, and so on. Peculiar fractal patterns are present also in crack nets that develop in dry earth or in old paintings under tensile stresses due to shrinkage.

For what concerns concrete, starting from stereology concepts Stroeven [18] pointed out the resolution dependence of damage parameters as a consequence of the power-law feature of the aggregate sieve curve. It is well-known, in fact, that the weakest link in concrete is represented by the particle–matrix bond. Thereby, the microcrack distribution inside material is governed by the aggregate one. In the case of the typical Füller distribution, it can be shown that the fractal dimension of the debonding cracks, along a straight line, is equal to 0.5.

Hence consider now the simplest structure, a bar subjected to tension, where, at the maximum load, dilation strain tends to concentrate into a fractal damage band, while the rest of the body undergoes elastic unloading. If, for the sake of simplicity, we assume that the strain (within the band) is localized onto cross-sections whose projection on the longitudinal axis is given by a deterministic fractal set, e.g. the Cantor set, the displacement function at rupture can be represented by a Devil's staircase graph. This last graph is a singular fractal function that is constant everywhere except at the points corresponding to the Cantor set (Fig. 4a). The strain defined in the classical manner is meaningless at the singularity points, since it tends to diverge even if the displacement discontinuities are finite. On the other hand, an original definition of fractal strain acting upon lacunar domains was put forward by Cornetti [12].

Let $\Delta_\varepsilon = 1 - d_\varepsilon = \ln 2/\ln 3 = 0.6391$ be the fractal dimension of the lacunar projection of the damaged sections. Since $\Delta_\varepsilon$ must be less than or equal to 1, the fractional decrement $d_\varepsilon$ is comprised between 0 and 1. One can compute the critical displacement $w_c$ as the product of the nominal strain at rupture $\varepsilon_c$ times the

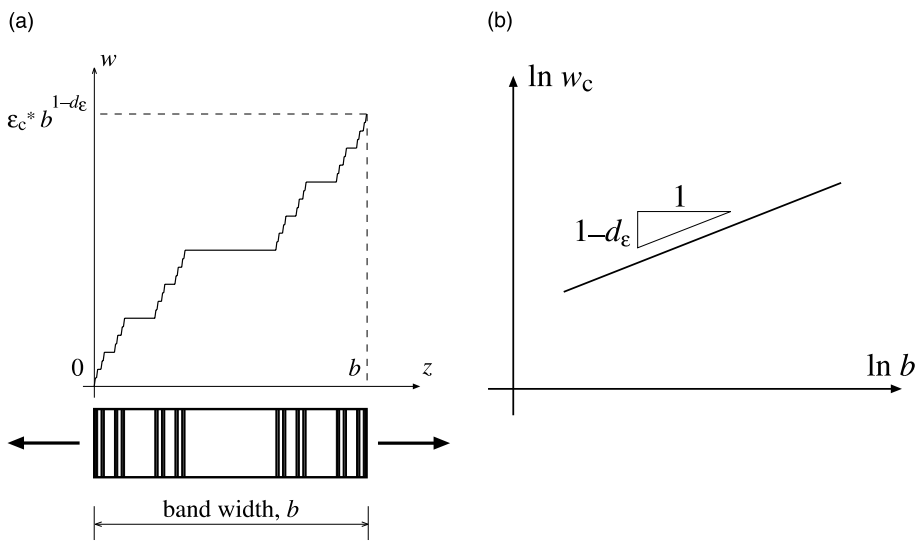


Fig. 4. Fractal strain localization (a) and scaling law for critical displacement $w_c$ (b).

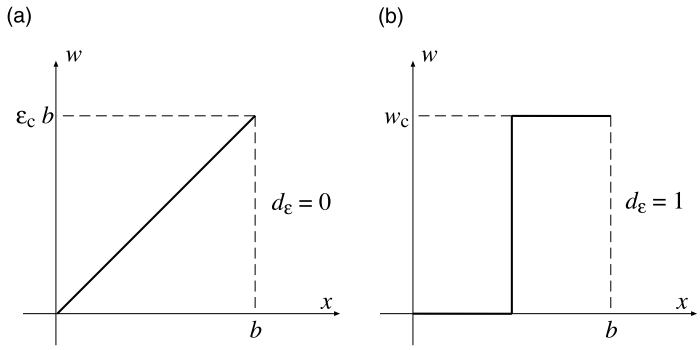


Fig. 5. Extremely diffused strain (a) and extremely localized deformation (b) over the bar.

nominal width $b$ of the damaged band. On the other hand, the Euclidean measure of the damaged band is scale dependent. Thus, according to the fractal measure of the damage line projection, the total elongation of the damage band at rupture must be given by the product of the fractal measure $b^{(1-d_\varepsilon)}$ of the Cantor set times the *fractal critical strain* $\varepsilon_c^*$:

$$w_c = \varepsilon_c b = \varepsilon_c^* b^{1-d_\varepsilon} \tag{3}$$

Due to dimensional homogeneity, the fractal critical strain $\varepsilon_c^*$ holds the anomalous physical dimension $[L]^{d_\varepsilon}$. The fractal critical strain is the true material constant, i.e. it is the only scale-invariant parameter governing the kinematics of the fractal band. The fractal scaling law of the critical displacement can be obtained from Eq. (3) and put into logarithmic form:

$$\ln w_c = \ln \varepsilon_c^* + (1 - d_\varepsilon) \ln b \tag{4}$$

which implies linear scaling in the bilogarithmic diagram, with slope equal to $1 - d_\varepsilon$ (Fig. 4b). If the size of the specimen is varied, Eq. (4) represents the positive size effect on the critical displacement $w_c$, typically revealed by tensile tests as documented by van Mier and van Vliet [5].

The fractional exponent $d_\varepsilon$ is intimately related to the degree of disorder in the mesoscopic damage process: the smaller $d_\varepsilon$, the higher the disorder occurring into the kinematics of damage. When $d_\varepsilon$ varies, the kinematical controlling parameter $\varepsilon_c^*$ moves from the classical critical strain $\varepsilon_c$ ($[L]^0$) to the critical crack opening displacement $w_c$ ($[L]^1$). Therefore, when $d_\varepsilon = 0$ (diffused damage, ductile behavior), one obtains the classical response, where the collapse is governed by the canonical critical strain $\varepsilon_c$, independently of the band size $b$ (Fig. 5a). In this case, continuum damage mechanics holds, and the critical displacement $w_c$ is subjected to the maximum scale-effect ($w_c \sim b$). On the other hand, when $d_\varepsilon = 1$ (localization of damage onto single cross-sections, i.e. brittle behavior, Fig. 5b) the collapse is governed by the critical displacement $w_c$, which is size-independent as in Hillerborg's formulation. This means that $d_\varepsilon = 1$ corresponds to Fracture Mechanics and to the most relevant structural embrittlement with the increase of structural size.

## 4. Fractal fracture energy: relation among the scaling exponents

Several experimental investigations [3,19] have shown that also the third mechanical quantity involved into the cohesive law, i.e. the fracture energy $\mathscr{G}_F$, is scale dependent. More precisely, $\mathscr{G}_F$ increases with the size of the specimen. This peculiar behavior can be explained by assuming that, after the peak load is reached, the energy is dissipated over the infinite number of lacunar sections where softening takes place inside the damage band. During the softening regime, the damage process is governed by the fractal stress

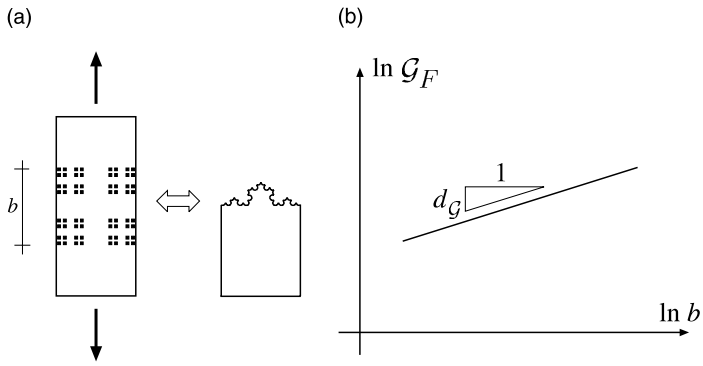


Fig. 6. Energy dissipation domain (a) and scaling law for fracture energy $\mathscr{G}_{\mathrm{F}}$ (b).

and strain, in such a way that we can generalize Eqs. (2) and (4) to any load and displacement values, obtaining:

$$\sigma = \sigma^* b^{-d_\sigma}, \quad w = \varepsilon^* b^{1-d_\varepsilon} \tag{5}$$

Differently from $\sigma_{\mathrm{u}}$ and $w_{\mathrm{c}}$, the fracture energy is an integral quantity defined over the whole softening regime, i.e. $\mathscr{G}_{\mathrm{F}} = \int_0^{w_{\mathrm{c}}} \sigma \, \mathrm{d}w$. When applied to this integral, the change of variables (5) allows us to highlight the effect of the structural size to the nominal fracture energy:

$$\mathscr{G}_{\mathrm{F}} = \int_0^{w} \sigma \, \mathrm{d}w = b^{1-d_\varepsilon-d_\sigma} \int_0^{\varepsilon_{\mathrm{c}}^*} \sigma^* \, \mathrm{d}\varepsilon^* = \mathscr{G}_{\mathrm{F}}^* \, b^{1-d_\varepsilon-d_\sigma} \tag{6}$$

The power law describing the scaling of the fracture energy (Fig. 6b) is provided by the fractality of the domain where energy is dissipated. According to the fractal model, the damage process takes place on a domain $A^*$ (different from the one of Eq. (1)) with a dimension $\Delta_{\mathscr{G}} = 2 + d_{\mathscr{G}}$ larger than 2. Instead, in the classical Hillerborg's model, the measure of the nominal smooth fracture surface was $A_0 = b^2$. Applying the renormalization procedure, the total energy expenditure $W$ necessary to take the specimen to failure is found to be equal to

$$W = \mathscr{G}_{\mathrm{F}} A_0 = \mathscr{G}_{\mathrm{F}}^* A^* \tag{7}$$

While $A_0$ and $\mathscr{G}_{\mathrm{F}}$ are only nominal or fictitious quantities, $A^*$, the *invasive* dissipation domain, and $\mathscr{G}_{\mathrm{F}}^*$, the *fractal fracture energy*, are, respectively, the true geometrical and physical variables of the process. $\mathscr{G}_{\mathrm{F}}^*$ possesses the anomalous physical dimensions $[\mathrm{FL}][\mathrm{L}]^{-(2+d_{\mathscr{G}})}$ and, as well as $\sigma_{\mathrm{u}}^*$ and $\varepsilon_{\mathrm{c}}^*$, is scale independent. Since $A^* \approx b^{(2+d_{\mathscr{G}})}$, the value of $d_{\mathscr{G}}$ can be obtained from Eqs. (6) and (7) as a function of $d_\sigma$ and $d_\varepsilon$. Hence, the fundamental relationship among the three scaling exponents is

$$d_\sigma + d_\varepsilon + d_{\mathscr{G}} = 1 \tag{8}$$

where all the exponents are positive and lower than (or equal to) 1. While $d_\varepsilon$ can attain all the values in the interval [0,1], $d_\sigma$ and $d_{\mathscr{G}}$ seem to be comprised between 0 and 1/2 (Brownian disorder). Eq. (8) puts a clear restriction to the maximum degree of disorder. More specifically, Eq. (8) confirms that the sum of $d_\sigma$ and $d_{\mathscr{G}}$ is always lower than 1, as previously asserted, through dimensional arguments, by Carpinteri [7].

Attempts to explain the size effect on fracture energy through fractal geometry were pursued in the past. Some of them justified the size effect assuming (and verifying through experiments) that rough fracture surfaces in quasi-brittle materials have a dimension larger than 2 (i.e. they are invasive fractal surfaces like Brownian surfaces, see, Carpinteri and Chiaia [11]). The same authors [20] explained the size effect also as a

consequence of fractal crack propagation (i.e. by defining a fractal stress-intensity factor $K_I^*$). Fractal crack branching was assumed to cause scale-effects by Ji et al. [21]. In all the above theories the domain of dissipation has a dimension comprised between 2 and 3. In other words, the cracking domain is supposed to be "more" than a surface and "less" than a volume. In order to depict schematically this domain according to our model, we can simply assume a deterministic Cantorian lacunarity to describe both strain localization ($\Delta_\varepsilon = \ln 2/\ln 3$), and void distribution on the cross-section, ($\Delta_\sigma = 1 + \ln 2/\ln 3$). In this way, energy is dissipated over a fractal domain (the so-called Cantor dust, Fig. 6a) given by the Cartesian product of the two lacunar sets, whose dimension is the sum of the dimensions of the factors:

$$\Delta_{\mathscr{G}} = \Delta_\sigma + \Delta_\varepsilon \tag{9}$$

which yields $\Delta_{\mathscr{G}} = 1 + 2\ln 2/\ln 3 = 1 + \ln 4/\ln 3$. Note that Eq. (9) corresponds exactly to Eq. (8). Moreover, the Cantor dust describing the damage band has the same measure and dimension, $b^{(1+\ln 4/\ln 3)}$, of the Von Koch curve representing, as an archetype, the fracture surface (Fig. 6a). This shows that the present model is consistent with the previous results obtained by Carpinteri and Chiaia [11] simply assuming self-similarity of the crack surfaces. In other words, the process of dissipation described by the *fractal band model* is equivalent to the generation of an invasive fracture surface.

## 5. Scale-independent cohesive law and experimental results

The original Hillerborg's model is based on the assumption that both the critical crack opening displacement $w_c$ and the ultimate strength $\sigma_u$ are independent of the structural size. Unfortunately, experiments show that this is not the case. Consequently, the classical cohesive law cannot be considered as a material property if a wide range of scales is considered.

To overcome this limitation, our model associates to the linear elastic law (Fig. 7a), valid for the undamaged material, the relationship between fractal stress and fractal strain, assuming that $\sigma_u^*$ and $\varepsilon_c^*$ are the true scale-independent parameters. From a technical point of view, it is interesting to note that the fractal fracture energy $\mathscr{G}_F^*$ can be obtained as the area below the softening fractal stress–strain diagram (Fig. 7b):

$$\mathscr{G}_F^* = \int_0^{\varepsilon_c^*} \sigma^* \, d\varepsilon^* \tag{10}$$

During the softening regime, i.e. when most of the dissipation occurs, $\sigma^*$ decreases from the maximum value $\sigma_u^*$ to 0, while $\varepsilon^*$ grows from 0 to $\varepsilon_c^*$. In the meantime, the non-damaged parts of the bar undergo elastic

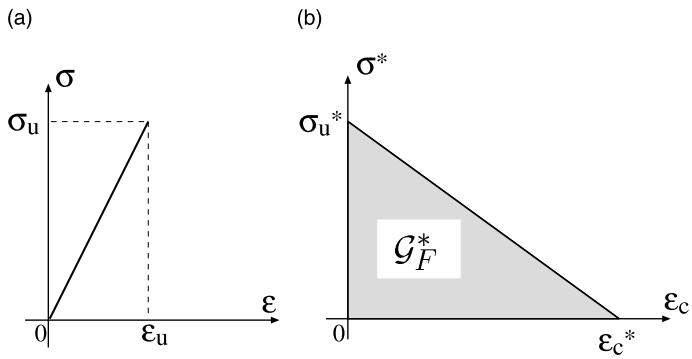


Fig. 7. Scale-independent cohesive crack model.

unloading. We call the $\sigma^*$–$\varepsilon^*$ diagram the *fractal* or *scale-independent cohesive law*, which is shown in Fig. 7b. Contrarily to the classical Hillerborg's law, which is sensible to the structural size, this curve is an exclusive property of the material.

Recently, van Mier and van Vliet [5] performed accurate tensile tests on dog-bone shaped specimens over a wide scale range (1:32). They plotted the cohesive law for specimens of different sizes. Except in the case of the smallest specimens (where other phenomena, like shrinkage, are assumed to bias the results) they found that, *with increasing the specimen size, the peak of the curve decreases whereas the tail rises*. That is, tensile strength decreases while critical displacement increases. More in detail, $w_c$ varies more rapidly than $\sigma_u$ does. Therefore, an increase of the area beneath the cohesive law, i.e. of the fracture energy, is observed. Thus, the experimental trends of $\sigma_u$, $\mathscr{G}_F$, $w_c$ confirm the conclusions of the fractal model.

In order to highlight the implications of the proposed model onto the nominal constitutive laws, we can suppose linearity to hold for both the $\sigma$–$\varepsilon$ and $\sigma^*$–$\varepsilon^*$ diagrams (Fig. 7). Consider now three specimens, made of the same material, presenting a ligament size respectively equal to $b_1$, $b_2 = 2b_1$, $b_3 = 4b_1$, and a band thickness proportional to the ligament. If $w_{c1}$, $\varepsilon_{c1}$, $\sigma_{u1}$ are the parameters of the first specimen, the $\sigma$–$\varepsilon$ and $\sigma$–$w$ relationships can be plotted (Fig. 8) for the three specimens, assuming, for instance, that $d_\sigma = 0.2$ and $d_\varepsilon = 0.6$ (hence $d_{\mathscr{G}} = 0.2$). Applying the fractal model one obtains, in the $\sigma$–$\varepsilon$ diagram (Fig. 8a), a more brittle behavior as the size increases, even if this trend is less pronounced than in Hillerborg's model.

The nominal cohesive laws, according to the fractal model, can be represented by three different curves (Fig. 8b), one for each specimen (remember that, instead, the fractal cohesive law is a unique scale-independent curve). Note that the tensile strength decreases, while the fracture energy and the critical displacement increase. This justifies the rising experimental tails observed by van Vliet and van Mier [5].

The model has been eventually compared with the uniaxial tension data obtained by Carpinteri and Ferro [3,4] on dog-bone shaped specimens (Fig. 9a) under fixed boundary conditions. They focused their attention on the size effect on the ultimate stress and on the fracture energy and interpreted their values by means of fractals. The exponents of the scaling laws were deduced by fitting the experimental results. In particular, they found the values $d_\sigma = 0.14$ and $d_{\mathscr{G}} = 0.38$. The $\sigma$–$\varepsilon$ and the $\sigma$–$w$ diagrams are reported in Fig. 9b and c. Here, $w$ is the displacement localized in the damage band, obtained by subtracting, from the total one, the displacement due to elastic and inelastic pre-peak deformation. Note that the trends are the same observed in the theoretical curves of Fig. 8. Last but not least, the value $d_\varepsilon = 0.48$, very near to the theoretical value 0.5 quoted in Section 3, is yielded by Eq. (8), so that the fractal cohesive law can be represented in Fig. 9d. As expected, all the curves related to the single sizes tend to merge in a unique, scale-independent cohesive law.

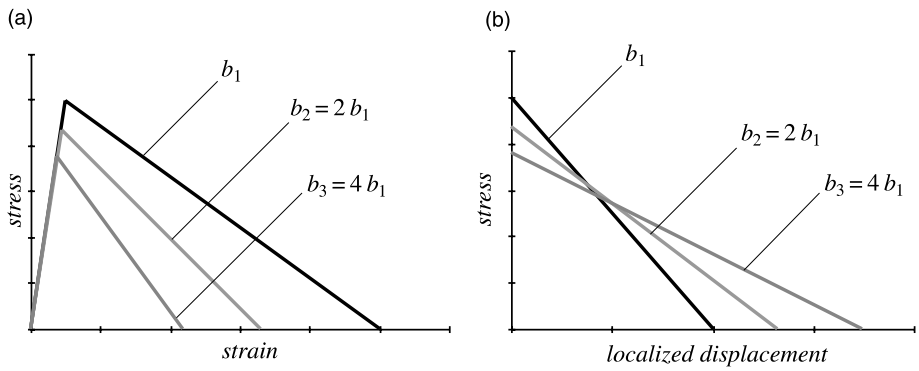


Fig. 8. Implications of the fractal cohesive law on the nominal $\sigma$–$\varepsilon$ (a) and $\sigma$–$w$ (b).

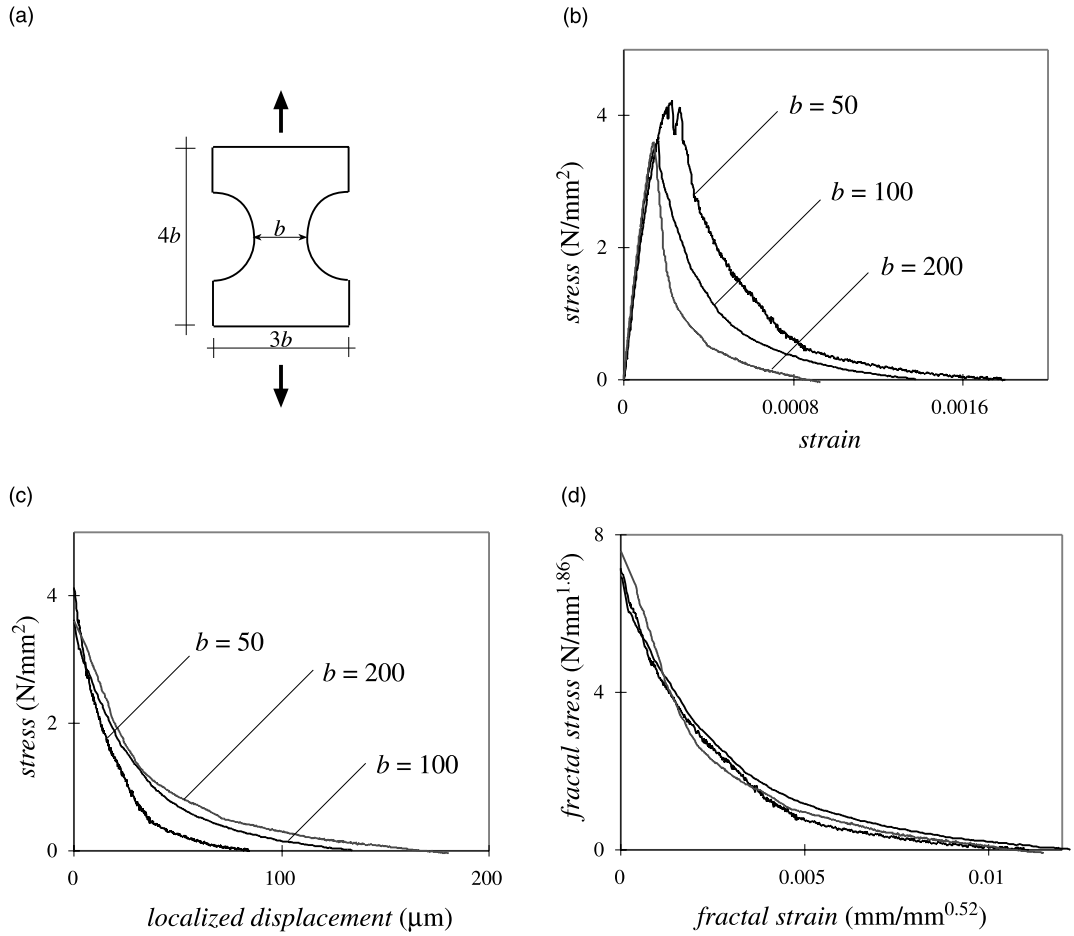


Fig. 9. Stress vs. strain (b), cohesive law (c), fractal cohesive law (d) for dog-bone specimens (a) of width $b$ (mm) [3].

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