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Static–kinematic duality and the principle of virtual work in the mechanics of fractal media

A. Carpinteri *, B. Chiaia, P. Cornetti

Department of Structural Engineering and Geotechnics, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

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Abstract

The framework for the mechanics of solids, deformable over fractal subsets, is outlined. While displacements and total energy maintain their canonical physical dimensions, renormalization group theory permits to define anomalous mechanical quantities with fractal dimensions, i.e., the fractal stress $[\sigma^*]$ and the fractal strain $[\varepsilon^*]$. A fundamental relation among the dimensions of these quantities and the Hausdorff dimension of the deformable subset is obtained. New mathematical operators are introduced to handle these quantities. In particular, classical fractional calculus fails to this purpose, whereas the recently proposed *local fractional operators* appear particularly suitable. The static and kinematic equations for fractal bodies are obtained, and the duality principle is shown to hold. Finally, an extension of the Gauss–Green theorem to fractional operators is proposed, which permits to demonstrate the Principle of Virtual Work for fractal media.

Keywords: Renormalization group transformations; Fractals; Anomalous dimensions; Fractional Calculus; Solid mechanics

1. Introduction

It is well known that, in many cases, macroscopic smooth deformations in solids are accompanied by step-wise mesostructural changes. Among the most typical examples of such irregular behaviour, we mention the Portevin–Le Chatelier effect in metals, the slip band formation in metals and rocks, the damage accumulation in concrete-like materials, and the stick–slip phenomena in the frictional contact of rough interfaces. These patterns, which at a first sight look completely random, reveal surprising symmetries at a closer scale of observation. Therefore, fractal geometry seems to be the natural instrument to deal with these phenomena. Indeed, experimental evidence of fractality in disordered materials (e.g., rough interfaces, microcracked or porous continua, etc.) has been detected by the authors over a broad range of scales [1].

Fractal sets are characterized, as a matter of fact, by complex morphology, dilation symmetry (self-similarity) and non-integer Hausdorff dimension [2]. Mechanical quantities have been defined until now under the assumption of integer physical dimensions. Early attempts to extend the definition of mechanical quantities to media with fractal boundaries were made by Panagiotopoulos [3,4]. His first approach made use of the so-called Iterated Functions Systems (IFS). In this theory, mechanical quantities retain their usual dimensions and thus the approach is restricted to a simple morphologic description of fractal bodies. Afterwards, Panagiotopoulos [4] tried to overcome the

* Corresponding author. Tel.: +39-11-5644850; fax: +39-11-5644899.
E-mail address: carpinteri@polito.it (A. Carpinteri).

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limitations of a heuristic approach like IFS, and moved to functional analysis in Besov spaces. In this context, Jonsson and Wallin [5] had shown that functionals usually defined in the Sobolev space, in the case of fractals can be re-defined in a particular Besov space, by applying the method of atomic decomposition. This method is physically compatible with the properties of the Besov space which, as the space of forces or stress in a fractal (spongy) body, contains infinite series of concentrated forces. Unfortunately, only general arguments were traced, due to the immature death of Panagiotopoulos.

Some problems of contact mechanics have been considered from the standpoint of parametric homogeneity by Borodich [6]. Parametric homogeneous functions exploit a kind of incomplete self-similarity in the sense of Barenblatt [7]. They extend the notion of self-similarity to model the scaling properties of natural phenomena. Interesting results have been achieved for fractal punching and stick–slip crack propagation.

Instead of considering functional theory, Carpinteri [8,9] drew his attention to physics, and explored the possibility of handling mechanical quantities on fractals by means of Renormalization Group (RG) transformations. RG developed in statistical physics, as a powerful tool to handle diverging quantities arising in quantum mechanics and statistical physics [7,10]. Scale-invariance requires that the mechanical laws must always be independent of scale and of the measuring units. This yields to symmetrical groups of transformations, usually controlled by canonical exponents in the case of Euclidean domains. In the case of fractal domains, more complicated transformation groups come into play, the asymptotic measures being finite only if the appropriate noninteger exponents are considered.

The purpose of RG methods is to extract macroscopic models from microscopic phenomena and to obtain the so-called *universal* properties, i.e., scale-invariant quantities. The price to pay is that the appropriate mechanical quantities (i.e., stress, strain and work of deformation) have to assume noninteger physical dimensions. Applying RG to the deformation and damage behaviour of fractal media consists in the definition of new mechanical quantities with physical dimensions depending on the fractal dimension of the microstructure. In addition, these quantities undergo a natural dimensional transition according to the scale at which they are measured. The classical Euclidean description becomes consistent only in the limit of the largest scales, when the characteristic length of the phenomenon is negligible with respect to the macroscopic size of the structure.

In a previous paper, Carpinteri and Chiaia [11] applied these concepts to the two basic local quantities governing the fracture behaviour of disordered materials, namely the tensile strength and the fracture energy. They showed that the nominal quantities, defined according to the classical Euclidean continuum, are scale-dependent and undergo relevant size-effects. Thus, two anomalous quantities were defined, the fractal tensile strength σ_u^* and the *fractal fracture energy* G_F^*, characterized by non-integer dimensions, representing the true material constants at the critical point. Cornetti [12] made a step forward, considering, instead of a single fracture cross-section, the whole volume of the band where damage spreads. In this way, RG is applied to the critical strain ε_c, obtaining the *fractal critical strain* ε_c^* whose scaling law completes and enriches the previous laws determined for strength and toughness. The transition from a strain-controlled collapse to a crack-opening-controlled collapse, as the size of the structure increases, was put into evidence.

In this paper, the RG procedure is extended to the whole deformation process. Therefore, three anomalous mechanical quantities can be defined, namely the fractal stress σ^*, the fractal strain ε^*, and the fractal unit work of deformation W^*. The fundamental relation among these quantities is investigated. This permits to obtain a universal equation among their fractal dimensions and, therefore, a restriction to disorder. New mathematical operators are introduced to handle these quantities. Classical fractional calculus fails to this purpose, whereas the recently introduced *local fractional operators* appear particularly suitable. The basic static and kinematic equations for fractal media are obtained, and the duality principle of mechanics is shown to hold. Finally, an extension of the Gauss–Green theorem to fractional quantities is put forward, which permits to demonstrate the Principle of Virtual Work for fractal media.

In the following, it is assumed that the reader is acquainted with the basic concepts of fractal geometry (i.e., Hausdorff dimensions and measures, non-differentiability, self-similarity). Excellent reference textbooks are the ones by Mandelbrot [13] and Falconer [2].

2. Mechanical quantities on fractal media

2.1. Renormalized work of deformation

Let us consider the archetype of fractal porous media. It is constructed starting from the solid cube of unit dimension (*initiator* of the fractal set), which is divided into 27 smaller cubes with side $r_1 = 1/3$. The six cubes at the centre of each side are removed as well as the central cube, so that the cube has square passages through the centre of each side. Twenty cubes with linear dimension r_1 remain, so that $N_1 = 20$.

At the second step, the same transformation is applied to each of the remaining 20 cubes, i.e., only $N_2 = 400$ cubes (out of 729) of linear size $r_2 = 1/9$ are maintained. The transformation can be applied recursively and indefinitely, so that in the limit the fractal set known as the Menger sponge is obtained (Fig. 1). The self-similarity relation permits to calculate the fractal dimension Δ_ω of the sponge as

$$\Delta_\omega = \frac{\log 20}{\log 3} \cong 2.727 \tag{1}$$

For such deterministic fractal sets the self-similarity dimension coincides with the Hausdorff dimension. Thus, the Menger sponge is a lacunar set because its Hausdorff dimension is smaller than the topological one ($D = 3$). It is worth to say, incidentally, that the use of Hausdorff dimensions, appropriate for idealized mathematical fractal sets, is impossible to apply in practice. A more treatable quantity is represented by the *box-counting* dimension [13], which has been used by the authors to evaluate the fractal dimension of real objects [1].

It is interesting to note that the density of spongy objects is not constant but decreases with size. If a cut-off resolution is fixed (e.g., the resolution capability of an optical instrument), the density and the porosity of sponges can be computed as the linear size increases (Fig. 2). If b_0 is the size of the smallest cube, its apparent density is ρ_0 and its porosity is zero. For the next larger sponge ($b_1 = 3b_0$), the apparent density ρ_1 is equal to $20\rho_0/27$ and its porosity is 7/27. In the case of the larger sponge, ($b_2 = 9b_0$), the porosity is 329/729 while the apparent density drops to $\rho_2 = 400\rho_0/729$. Generalizing to sponges of arbitrary size b, the following scaling relations are provided, respectively, for porosity and apparent density:

$$\phi = 1 - \left[\frac{b_0}{b}\right]^{3-\Delta_\omega}, \tag{2a}$$

$$\rho = \rho_0 \left[\frac{b_0}{b}\right]^{3-\Delta_\omega}. \tag{2b}$$

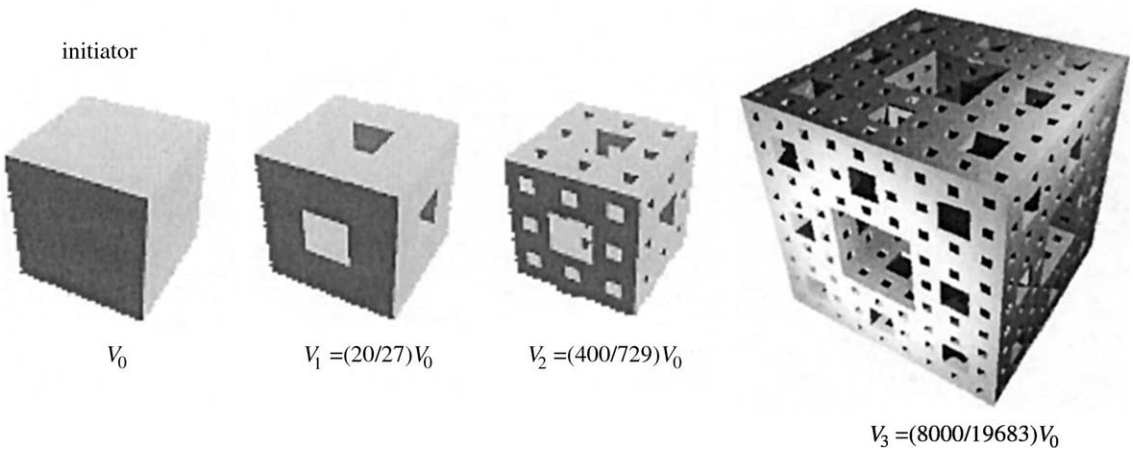

Fig. 1. Self-similar construction of the Menger sponge ($\Delta_\omega = 2.727$). As the resolution increases, the nominal volume decreases and, correspondingly, the fictitious work of deformation per unit volume, W_n, must increase to balance the work done by external forces. Therefore, if b is a linear size of the sponge, its scale-invariant measure is $b^{2.727}$. Accordingly, renormalization group yields the work of deformation W^* per unit of fractal volume, with dimensions $[\mathrm{F}][\mathrm{L}]^{-1.727}$.

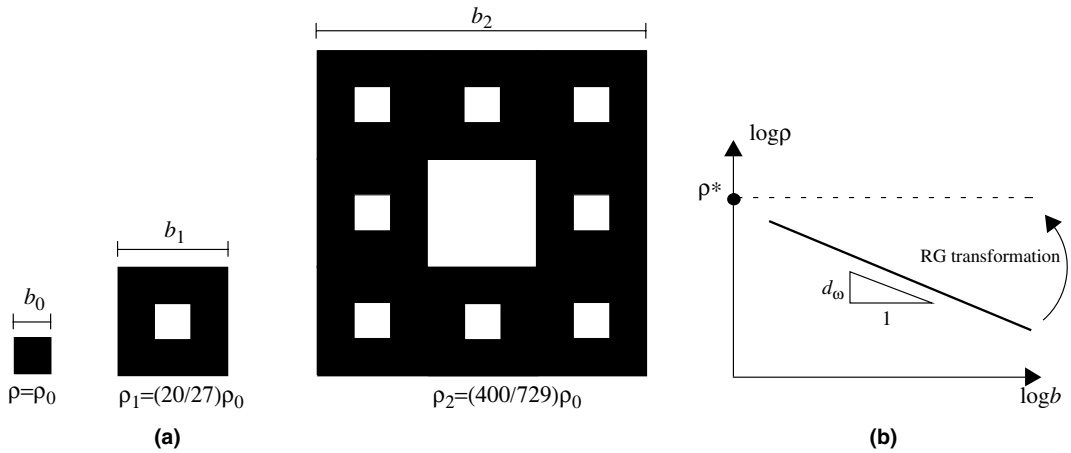

Fig. 2. Front view, at the same resolution, of Menger sponges with different sizes (a). The nominal density of sponges ρ decreases with size according to a power-law (b). This means that the probability to meet larger voids in real porous materials increases self-similarly. Renormalization group permits to define the scale-invariant density ρ^* with anomalous physical dimensions $[\mathrm{M}][\mathrm{L}]^{-(3-d_\omega)}$.

While the first is not a power-law relation, the second relation (Fig. 2(b)) displays typical fractal behaviour. These equations confirm the systematic experimental discovery that the probability of finding larger voids and defects into porous materials increases with size.

Fractal media are thus topologically intermediate between surfaces and volumes. For instance, in the asymptotic limit of the smallest resolutions, the Menger sponge is made of a 3D array of monodimensional fibres connected at singular points. Thus, the Euclidean measure (volume) of the pores is finite, and equals the total volume of the object, whereas the volume of the solid part is zero. These sets are *not measurable by means of the canonical dimensions*. Finite measures can be obtained only by means of noninteger dimensions, i.e., $[\mathrm{L}]^{2.727}$. Generalizing to any lacunar body, the fractal dimension can be expressed as $\Delta_\omega = 3 - d_\omega$. The fractional exponent d_ω controls the scaling of the mass density, according to Eq. (2b): $\rho \sim b^{-d_\omega}$. The larger d_ω, the higher the lacunarity of the body.

As shown in Fig. 1, where the fractal is represented at various magnification scales, refining the scale of observation, the body, at each step, possesses a smaller amount of nominal volume. At level zero (macrolevel) the domain looks like a compact body (volume $= V_0$) and the work of deformation can be stored in the whole body. At level 1 it can be easily noticed that the volume is reduced to $V_1 = (20/27)V_0 < V_0$, and so on, V_n ideally tending to zero in the limit represented by the proper fractal set (V_∞). This self-similar weakening is due to the presence of pores, voids and cracks at all length scales, and its extent depends also on the evolution of material damage.

Suppose that the body is deformable only over a fractal lacunar subset. Let the body be deformed, e.g., under the action of external forces, and let us neglect, at a first step, any kind of energy dissipation. To each scale of observation n, a nominal work of deformation per unit volume, W_n, can therefore be associated. On the other hand, the total stored energy L is a macroparameter, independent of the observation scale, since it has to balance the work done by the external forces. Therefore, considering a sequence of scales of observation, the following renormalization group transformations hold:

$$L = W_0 V_0 = W_1 V_1 = W_2 V_2 = \cdots = W_n V_n = \cdots = W^* \Omega^*, \tag{3}$$

where Ω^* is the Hausdorff measure of the deformable fractal subset, with dimensions equal to $[\mathrm{L}]^{\Delta\omega}$. Due to dimensional homogeneity, the fractal work of deformation W^* has the anomalous dimensions $[\mathrm{F}][\mathrm{L}]^{-(\Delta\omega-1)}$. The fractal work of deformation W^* is a true material property, i.e., it is scale-invariant.

2.2. Renormalized stress

In solid mechanics, we are concerned with the way by which forces are transmitted through the medium. The Cauchy definition of stress tensor $[\sigma]$ relies on some regularity properties (continuity and measurability)

of the medium. The concept of stress belongs to those physical models for which a direct measure is not possible on the basis of their definition. The stress, by its nature, is set apart from direct experience and is therefore a mental construct.

If a macroscopic description of the stress field is desired, classical mechanics is sufficiently accurate. On the other hand, when the microstructure of the body plays a fundamental role, a mesoscopic approach is required. For instance, in the case of biological tissues, the spatial organization of the constituents requires a multi-scale approach. The spatial hierarchy of collagen fibres in the human tendon is responsible for its remarkable viscoelastic properties. Compact bones can be treated as a composite material and their mechanical properties depend not only on the composition but also on the geometric structure. A mesoscopic model should be considered also for isotropic media in the presence of strain localization and large stress gradients (e.g., fracture and contact problems). In addition, defects are present *at all scales* in engineering materials and interact with each other in a complex manner. The synergetic properties of disordered distributions of microcracks provide peculiar behaviour under increasing strain. Attempts to describe these phenomena by classical concepts are incomplete or even misleading.

A consistent modelization of the stress flux through porous media can be pursued by means of fractal geometry. From this point of view, the rarefied cross-section on which stress is to be defined can be modelled by a *lacunar* fractal set of dimension Δ_σ, with $\Delta_\sigma \leqslant 2$. The lacunarity of the stress-carrying cross-section is provided by the dimensional decrement d_σ with respect to the Euclidean dimensionality $D = 2.0$, that is, $\Delta_\sigma = 2 - d_\sigma$. The probability to meet a large void becomes higher in larger domains. As already shown in Section 2.1, the apparent Euclidean measure (length, area or volume) of lacunar sets is scale-dependent and tends to zero as the resolution increases (Fig. 3). Thus, the Cauchy definition of stress cannot be applied. Instead, an original definition of fractal stress acting upon lacunar domains was put forward by Carpinteri [8] by applying the RG procedure to the nominal stress tensor $[\sigma]$. For the sake of simplicity, uniaxial stress fields will be considered, in order to deal with scalar relations. Extension to multiaxial stress fields is straightforward.

Considering, for instance, an orthogonal cross-section of the Menger sponge (Fig. 1), a lacunar set in the plane is obtained, which is called the *Sierpinski carpet* (Fig. 3). The fractal dimension of this domain is $\Delta_\sigma = \log 8/\log 3 \cong 1.893$ ($d_\sigma = 0.107$). As shown in Fig. 3, where the fractal domain is represented at various magnification scales, refining the scale of observation, the resisting cross-section, at each step, is represented by a smaller amount of nominal area. At level zero (macrolevel) the domain looks like a compact surface ($A_0 =$ area) and the whole resisting section appears to transmit the stresses through the

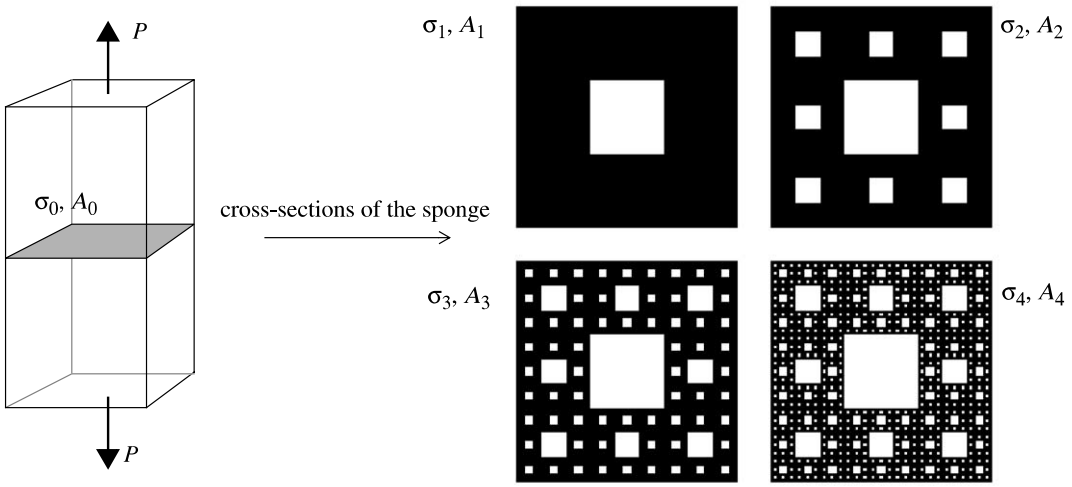

Fig. 3. Renormalization group for the nominal stress over a Sierpinski carpet ($\Delta_\sigma = 1.893$). The fractal stress σ^*, with physical dimensions $[\mathrm{F}][\mathrm{L}]^{-1.893}$, is defined at any singular point of the fractal set as: $\lim_{\Delta A^* \to 0}(\Delta P/\Delta A^*)$.

body. At level 1 it can be easily noticed that the resisting section is reduced to $A_1 = (8/9)A_0 < A_0$, while at level 2 it becomes $A_2 = (64/81)A_0 < A_1$ and so on, A_n ideally tending to zero in the limit represented by the proper fractal set ($A_\infty = A^*$).

To each scale of observation, a fictitious microstress σ_n is associated, which can be considered as the stress carried at the scale n. On the other hand, the total external force P is independent of the observation scale, since it has to fulfill global equilibrium. Therefore, a sequence of scales of observation can be considered, where the first scale is the macroscopic one (A_0 being the nominal cross-sectional area and σ_0 the conventional tensile stress), and the asymptotic scale of observation is the microscopic (fractal) one ($A_\infty = A^*$ being the *measure*, in the Hausdorff sense, of the fractal set and σ^* the so-called *fractal stress*). The following RG holds:

$$P = \sigma_0 A_0 = \sigma_1 A_1 = \sigma_2 A_2 = \cdots = \sigma_n A_n = \cdots = \sigma^* A^*. \tag{4}$$

Due to dimensional homogeneity, if $[\mathrm{L}]^{(2-d_\sigma)}$ is the dimension of the fractal cross-section, σ^* has the anomalous dimensions $[\mathrm{F}][\mathrm{L}]^{-(2-d_\sigma)}$. The fractal stress σ^* is the scale-invariant mechanical parameter. It is worth to notice that, as in the case of (3), the above procedure permits to define an *effective* mechanical quantity, i.e., a mean fractal stress acting upon the domain A^*. Instead, in order to obtain a local definition of σ^*, exactly as in the case of the definition of the classical Cauchy stress, the limit

$$\lim_{\Delta A^* \to 0} (\Delta P / \Delta A^*), \tag{5}$$

is supposed to exist and, eventually, to attain finite values at any singular point of the support A^*. This is mathematically possible for lacunar sets like that in Fig. 3 (and also for rarefied point sets like Cantor sets) which, although not compact, is dense in the surrounding of any singular point.

Choosing b as a characteristic size of the cross-section, one obtains: $A_0 \approx b^2$ and $A^* \approx b^{(2-d_\sigma)}$. Thus, equating the extreme members of the group (4), the following scaling law is obtained:

$$\sigma = \sigma^* b^{-d_\sigma}. \tag{6}$$

In a logarithmic form one obtains: $\log \sigma = \log \sigma^* - d_\sigma \log b$. This implies linear scaling in the bilogarithmic diagram, with slope equal to $-d_\sigma$ (Fig. 4). If attention is focused to the peak load P_u, the ultimate tensile stress σ_u must be considered, and Eq. (6) represents the negative size effect on tensile strength, experimentally revealed by several authors [8,9].

2.3. Renormalized strain

Continuum mechanics is based on the so-called static–kinematic duality [14]. This duality, which implies (and is implied by) the Principle of Virtual Work, intimately connects the static and kinematic quantities, independently of the material and geometry of the body. In the most general case of a three-dimensional deformable solid, duality relates the stress tensor $[\sigma]$ with the strain tensor $[\varepsilon]$, so to define the internal work

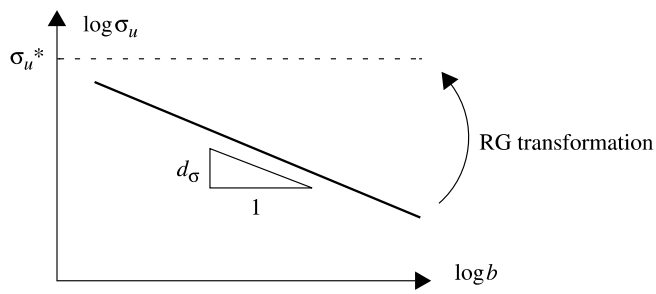

Fig. 4. A remarkable effect of the fractal scaling of the mechanical stress is the size-effect law for the nominal tensile strength. Experimental tests on disordered materials, carried out with any specimen geometry, unequivocally testify that the nominal strength σ_u decreases with increasing the size of the specimens. Instead, renormalization group yields the fractal strength σ_u^* as the true scale-invariant parameter.

of deformation. In the present section, the kinematical counterpart of the fractal stress $[\sigma^*]$, defined in the previous section, will be introduced: the *fractal strain* $[\varepsilon^*]$.

The starting assumption of the model [12] is that the overall displacement field of the body can be originated, at the microlevel, by very different mechanisms. The classical hypothesis of not allowing displacement discontinuities is removed. The discontinuities, in fact, can be localized on an infinite number of cross-sections, spreading throughout the body. The spatial distribution of these cross-sections can vary considerably, and may possess particular symmetries (self-similarity). It is worth to stress, here, that we are considering a generic body which is *deformable over a fractal subset*, i.e., the material itself does not need to have a fractal distribution, but its deformation field does. This is the case, for instance, of cementitious composites, which usually develop strain localization in some bands, especially concentrated at the weak aggregate/matrix interfaces.

The hypothesis of fractal deformation fields, although rather anomalous, is not a simple abstraction. In fact, different experimental investigations confirm the fractal character of deformation (Fig. 5), at least close to the critical point. In the bulk of some metals, when subjected to tension, the so-called *slip lines* develop with a typical fractal structure [15] resembling the lacunar characteristics of Saturn's rings. Plastic shear bands in highly stressed rock masses also display fractal patterns [16]. Any damage band, when observed at sufficiently high resolution, is made out of several smaller bands, which, in their turn, appear to be constituted by smaller and smaller bands, and so on. Kleiser and Bocek [17] investigated the formation of the slip lines inside copper (Fig. 5), computing both their spatial distribution and their amplitude. They also calculated the fractal dimension of the lines projection over an orthogonal segment (Fig. 6), finding the value $\Delta_\varepsilon \cong 0.52$.

Considering the simplest uniaxial model, a slender bar subjected to tension, it can be argued that the projection (over the horizontal axis) of the cross-sections where deformation localizes, is a lacunar fractal

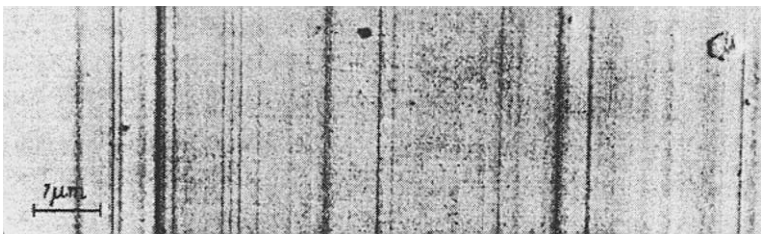

Fig. 5. Experimental slip lines in copper [17] at the micron scale. Note that the chaotic deformation pattern is self-similar and each slip band is constituted by smaller bands at a finer scale.

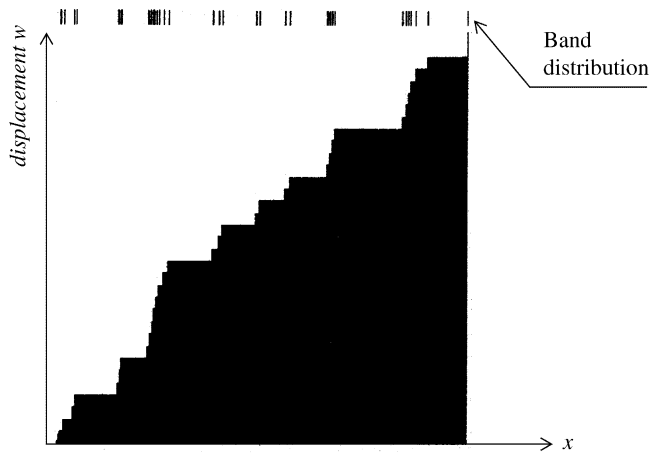

Fig. 6. Slip lines distribution and width [17]. The displacement w is a singular fractal function which is constant everywhere except over a fractal set of zero Lebesgue measure.

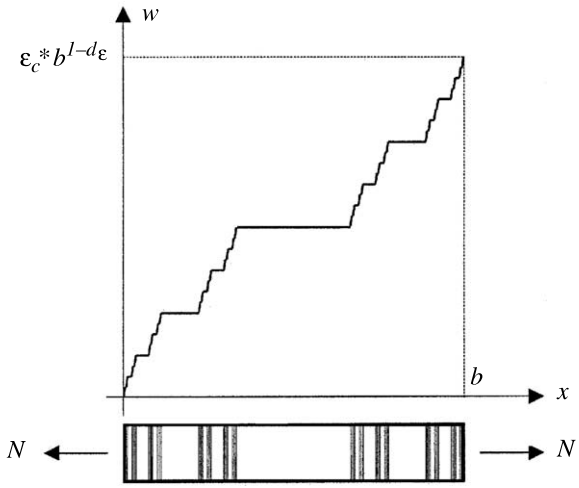

Fig. 7. Fractal localization of deformation on the stretched Cantor bar and singular fractal displacement function.

set, with dimension between zero and one. If the Cantor set ($\Delta_\varepsilon \cong 0.631$) is assumed as an archetype of the damage distribution, we may speak of the *fractal Cantor bar*, depicted in Fig. 7.

When fractal localization occurs inside the loaded bar, it is clear that the dilation strain tends to concentrate into singular stretched regions, while the rest of the body is practically undeformed. Thus, the displacement function can be represented by a *devil's staircase* graph, that is, by a singular fractal function which is constant everywhere except at the points corresponding to a lacunar fractal set of zero Lebesgue measure (Fig. 7).

Let $\Delta_\varepsilon = 1 - d_\varepsilon$ be the fractal dimension of the lacunar projection of the deformed sections. Since $\Delta_\varepsilon \leqslant 1$, the fractional decrement d_ε is always a number between 0.0 (corresponding to an Euclidean support, i.e., to strain fully smeared along the bar) and 1.0 (corresponding to the maximum localization of strain, i.e., to localized fracture surfaces). Similarly to the case of tensile stress, the RG procedure can be applied to the nominal strain ε developed inside the bar

$$w = \varepsilon_0 b_0 = \varepsilon_1 b_1 = \varepsilon_2 b_2 = \cdots = \varepsilon_n b_n = \cdots = \varepsilon^* b_0^{1-d_\varepsilon}, \tag{7}$$

where w is the total elongation of the bar. At the macroscopic scale, the nominal strain ε_0 is assumed to be constant throughout the bar. Thereby, the displacement w is given by the product of the canonical strain ε_0 times the length b_0 of the bar. By increasing the resolution, strain appears to be concentrated into smaller and smaller regions ($b_i < b_{i-1}$), as depicted in Fig. 8. Since, for a given bar, the total elongation is reso-

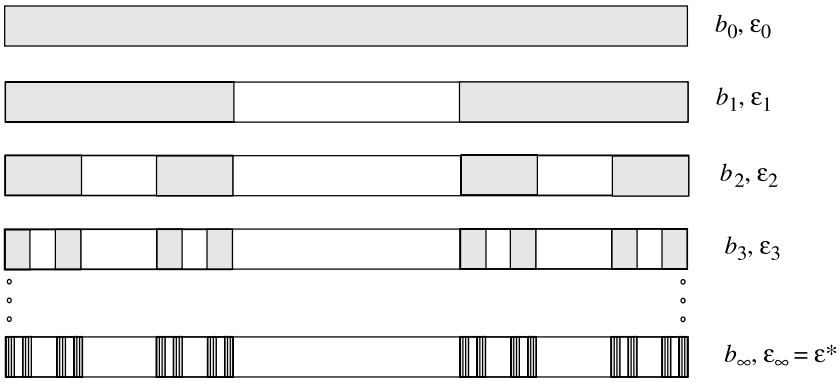

Fig. 8. Renormalization group for the tensile strain over a Cantor bar. The fractal strain ε^*, with dimensions $[L]^{0.369}$, is defined at any singular point of the Cantor set ($\Delta_\varepsilon = 0.631$) generated by the projection of the infinite slip lines where deformation is localized.

lution-independent, the product $\varepsilon_i b_i$ must be constant, and thus $\varepsilon_i \to \infty$ for $b_i \to 0$. In the limit of the finest resolution ($i \to \infty$), the strain is concentrated on an infinite number of sections, with zero total measure. According to the renormalization procedure, in order to overcome the presence of diverging and vanishing terms and to keep dimensional homogeneity, the microscale description requires the product of an anomalous (fractal) strain ε^* times the fractal measure $b_0^{1-d_\varepsilon}$ of the support.

The fractal strain ε^* is the only scale-independent parameter governing the kinematics of the fractal bar. Its physical dimensionality $[\mathrm{L}]^{d_\varepsilon}$ is intermediate between that of a displacement [L] and that of a pure strain $[\mathrm{L}]^0$ and synthesizes the conceptual transition between classical continuum mechanics ($d_\varepsilon = 0$) and fracture mechanics ($d_\varepsilon = 1$). The two limit situations are shown in Fig. 9, the devil's staircase being an intermediate situation with $d_\varepsilon \cong 0.369$.

If the critical situation is considered, equating the extreme members of the cascade Eq. (7), the following scaling relation for the nominal critical strain ε_c is obtained:

$$\varepsilon_c = \varepsilon_c^* b^{-d_\varepsilon}. \tag{8}$$

This corresponds to $\log \varepsilon_c = \log \varepsilon_c^* - d_\varepsilon \log b$, implying linear scaling in the bilogarithmic diagram, with slope equal to $-d_\varepsilon$. By varying the bar length, Eq. (8) provides the monofractal scaling law for the nominal critical strain ε_c, which is shown in Fig. 10.

The fractional exponent d_ε, representing the slope of the scaling law, is intimately related to the degree of disorder in the deformation process. By varying its value (e.g., for different materials or different loading levels), the transition from classical continuum damage mechanics to fracture mechanics is obtained. Correspondingly, the kinematical controlling parameter changes, from the nominal strain ε, to the crack opening displacement w.

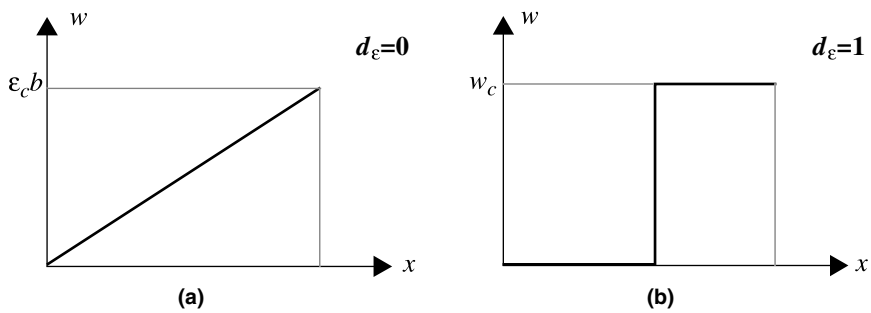

Fig. 9. Extremely diffused strain (a) and extremely localized deformation (b) over the bar. While the first case represents the classical homogeneous strain field, (evolving, with load, to smeared damage), the second diagram depicts a single displacement discontinuity, e.g., the formation of a sharp fracture.

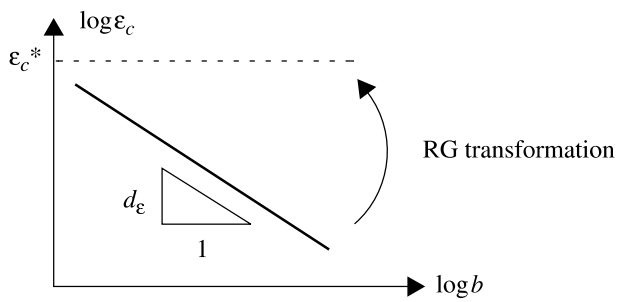

Fig. 10. Monofractal scaling law for the nominal critical strain ε_c. Renormalization group yields the fractal critical strain ε_c^* as the true scale-invariant parameter. The slope of the scaling law depends on the fractal dimension of the deformation pattern. In the classical case of homogeneous strain ($d_\varepsilon = 0$), no size-effect is revealed, whereas, when Fracture Mechanics rules the phenomenon ($d_\varepsilon = 1$), the embrittlement with size is sensible.

The smaller d_ε, the higher the disorder occurring into the kinematics of damage. When $d_\varepsilon = 0$ (diffused damage, ductile behaviour), one obtains the classical smooth behaviour, where the collapse is governed by the canonical critical strain ε_c, independently of the bar length. In this case, continuum damage mechanics holds (Fig. 9(a)), and the critical displacement w_c is subjected to the maximum size-effect ($w_c \sim b$). On the other hand, when $d_\varepsilon = 1$ (localization of damage on single cross-sections, i.e., brittle behaviour, see Fig. 9(b)), the collapse is governed by the critical displacement w_c, which is size-independent. In this case, the critical nominal strain ε_c undergoes a relevant negative size-effect. This means that $d_\varepsilon = 1$ corresponds to Fracture Mechanics and to the most relevant structural embrittlement with the increase of structural size.

3. Relation among the fractal exponents: universal bound to disorder

The definition of universal (scale-independent) mechanical quantities has been obtained at the price of losing the canonical physical dimensions. It is remarkable to notice that the three anomalous dimensions (i.e., d_ω, d_σ and d_ε) are not independent of each other, but intimately connected by a universal relation. As will be explained in the following sections, this connection implies the universal duality principle between statics and kinematics or, in other words, the Principle of Virtual Work for fractal media.

The fundamental relation among the fractional exponents can be easily obtained, in the case of the simple uniaxial model, by means of dimensional analysis arguments. It is physically reasonable to assume, in fact, that the work of deformation W^*, stored in the fractally deformable body, is produced within the infinite lacunar cross-sections where strain is localized. Thus, the fractal domain Ω^*, with dimension $3 - d_\omega$, where the strain energy is stored, must be equal to the Cartesian product of the lacunar fractal cross-sections with dimension $2 - d_\sigma$, times their cantorian projection with dimension $1 - d_\varepsilon$. Since the dimension of the Cartesian product of two fractal sets is equal to the sum of their dimensions [2], one obtains

$$(3 - d_\omega) = (2 - d\sigma) + (1 - d_\varepsilon), \tag{9}$$

which yields the fundamental relation among the exponents as

$$d_\omega = d_\sigma + d_\varepsilon. \tag{10}$$

The above relation, as will be shown in Section 6, yields important physical consequences. It means that the fractional exponents governing kinematics, statics and energy storage during the deformation process are not independent of each other, but obey a mutual relationship. Note that Euclidean media represent a particular case of Eq. (10), where $d_\omega = d_\varepsilon = d_\sigma = 0$.

In another paper [18], attention is drawn only to the critical point of rupture, and the dissipated energy G_F^* is considered instead of the work of deformation. Fractal strain localization is associated with energy dissipation, ranging from homogeneous dissipation (plasticity-like damage) when $d_\varepsilon = 0$, to highly localized dissipation (i.e., fracture mechanics) when $d_\varepsilon = 1$. The dimension of the dissipation domain is put equal to $2 + d_G$. Correspondingly, when $d_G = 1$ one obtains volume dissipation (plasticity). Instead, for $d_G = 0$ energy is dissipated on a sharp surface, as predicted by linear elastic fracture mechanics. In any intermediate case, the fractal domain, with dimension $2 + d_G$, where energy dissipation occurs, must be equal to the Cartesian product of the lacunar cross-sections, with dimension $2 - d_\sigma$, times the cantorian set where the critical strain is localized, with dimension $1 - d_\varepsilon$. The following relation can be obtained:

$$d_\varepsilon + d_\sigma + d_G = 1, \tag{11}$$

which is completely equivalent to Eq. (10), if one assumes $2 + d_G = 3 - d_\omega$.

4. Local fractional operators

In order to find mathematical tools suitable to work with functions and variables defined upon fractal domains, researchers started to examine the possibility of applying fractional operators, i.e., derivatives and integrals of noninteger order. The application of fractional operators should provide quantities charac-

terized by the requested noninteger physical dimensions and by peculiar scaling properties. However, despite these clear conceptual relations between fractional calculus and fractal geometry, a systematic link has not been found yet.

The concept of noninteger (fractional) differentiation is not new. In a letter to L'Hospital dated in 1695, Leibniz mentioned the 1/2-order derivative, conjecturing that, one day, it would have been usefully employed in Physics. Since then, the so-called "Fractional Calculus", that is, the study of mathematical operators able to make derivatives and integrals of *any order* (not necessarily integer), has developed in pure mathematics [19]. The concept of *dimensional continuity* clearly emerges in Von Neumann algebra, and mainly in the idea of dimensionally noninteger topological spaces, studied by Hausdorff at the beginning of the century and successively reconsidered and classified by Mandelbrot [13] as fractal sets. The earlier applications of the fractional calculus in Physics to be mentioned are those due to Heaviside [20] in electromagnetism and to Scott Blair [21] in rheology. Later on, Caputo and Mainardi [22] and Bagley [23], applied fractional operators to the rheological viscoelastic behaviour of materials.

In all the aforementioned applications, the fundamental aspect of fractional calculus is the capacity of continuously interpolating between the extreme situations of a physical phenomenon covering, in a concise and elegant manner, its whole constitutive spectrum. Consider, for instance, the scalar relation

$$\sigma = k\frac{\mathrm{d}^{\alpha}\varepsilon}{\mathrm{d}t^{\alpha}}. \tag{12}$$

This equation allows the constitutive description of a material possessing both elastic and viscous properties. In fact, if $\alpha = 0$ then $k = E([\mathrm{F}][\mathrm{L}]^{-2})$ and one obtains the Hooke's law for a linear elastic solid ($\sigma = E\varepsilon$). Instead, if $\alpha = 1$ then $k = \mu([\mathrm{F}][\mathrm{L}]^{-2}[\mathrm{T}])$ and one obtains the Newton's law for a viscous fluid ($\sigma = \mu\dot{\varepsilon}$). Therefore, the whole range of rheological behaviour (i.e., polymeric behaviour) can be simply described by varying the noninteger order of derivation α, recalling that the physical dimensions of the parameter k have to be accordingly modified.

The classical definitions of Fractional Calculus are due to Riemann and Liouville [24]. Their definition of fractional integral can be seen as a straightforward generalization of the Cauchy integral formula. Let α be a positive real number, f a continuous function of x, and let a be any fixed number. By definition, the "α-order integral" of $f(x)$ is given by ($\alpha > 0$)

$$\frac{\mathrm{d}^{-\alpha}f(x)}{[\mathrm{d}(x-a)]^{-\alpha}} = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(y)}{(x-y)^{1-\alpha}}\,\mathrm{d}y. \tag{13}$$

If $a = -\infty$ the above expression is usually referred to as the "Weyl integral". Based on Eq. (13), the α-order derivative of f can be obtained as the classical n-order derivative of the $(n-\alpha)$-order integral of f, where $n > \alpha$. This definition is independent of n, provided that $n > \alpha$. Thus, n is usually set as the smallest integer number larger than α. The most interesting case is given by $0 < \alpha < 1$, which evidently yields

$$\frac{\mathrm{d}^{\alpha}f(x)}{[\mathrm{d}(x-a)]^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{f(y)}{(x-y)^{\alpha}}\,\mathrm{d}y. \tag{14}$$

Another equivalent definition of the *differintegral* fractional operator, particularly useful for numerical implementation, is due to Grünwald and Leitnikov. For this definition, as well as for other details, the reader is addressed to the reference books by Oldham and Spanier [19], Miller and Ross [25], Samko et al. [26] and Podlubny [27]. As can be immediately deduced from Eq. (14), the α-order fractional derivative of a continuous function $f(x)$ depends on the lower integration limit a, except when α is a positive integer. This is the most important distinction between conventional and fractional differentiation. While, in fact, the classical derivative is a "local" operator, univocally determined in a point x, the fractional derivative possesses a "nonlocal" or "integral" character (it is, actually, an integral operator). It depends, in fact, on all the $f(y)$ values for $y \in [a, x]$, and not only on the value of the function in x and in its infinitesimal neighbourhood.

Among the few attempts to connect classical fractional calculus with fractal geometry, it is worth to mention those of Tricot [28], who related the Hölderian properties of a fractal graph to the maximum order of fractional derivation, Tatom [29], who pointed out the possibility of changing the Hausdorff dimensions

of fractal quantities by means of fractional differintegration, and Schellnhuber and Seyler [30], who explored the fractional differentiation on the devil's staircase.

Based on the classical definitions, Kolwankar [31] has recently introduced a new operator called "local fractional integral" or *fractal integral*. Let $[x_i, x_{i+1}],\ i = 0, \ldots, N-1,\ x_0 = a,\ x_N = b$, be a partition of the interval $[a, b]$, and x_i^* some suitable point of the interval $[x_i, x_{i+1}]$. Next, consider a function $f(x)$ defined upon a lacunar fractal set belonging to $[a, b]$. The fractal integral of order α of the function $f(x)$ over the interval $[a, b]$ is defined as

$$I^{\alpha}[f(x)]_a^b = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(x_i^*) \frac{\mathrm{d}^{-\alpha} 1_{\mathrm{d}x_i}(x)}{[\mathrm{d}(x_{i+1} - x_i)]^{-\alpha}}, \tag{15}$$

where $1_{\mathrm{d}x_i}(x)$ is the unit function defined upon $[x_i, x_{i+1}]$. The fractal integral is a mathematical tool suitable for the computation of fractal measures. In fact, it yields finite values of the measure if and only if the order of integration is equal to the dimension of the fractal support of function $f(x)$. Otherwise, its value is zero or infinite, thus showing a behaviour analogous to the Hausdorff measure of a fractal set.

Even more interesting for our purposes, based on Eq. (14), Kolwankar and Gangal [32,33] introduced the local fractional derivative (LFD) of order α, whose definition is $(0 < \alpha < 1)$

$$D^{\alpha} f(y) = \lim_{x\to y} \frac{\mathrm{d}^{\alpha}[f(x) - f(y)]}{\mathrm{d}(x-y)^{\alpha}}. \tag{16}$$

Differently from Eq. (14), the LFD is a function only of the $f(x)$ values in the neighbourhood of the point y where it is calculated. It has been proven [28] that the classical fractional derivative (Eq. (14)) of a non-differentiable (fractal) function (e.g., like the Weierstrass function) exists as long as its order is less than the Hölder exponent characterizing the singularity. Kolwankar [31] proved that, in the singular points, the LFD (Eq. (16)) is generally zero or infinite. It assumes a finite value only if the order α of derivation is equal to the Hölder exponent of the graph. For instance, in the case of the well-known devil's staircase graph (Fig. 7) the LFD of order $\alpha = \log 2/\log 3$ (i.e., equal to the dimension of the underlying middle-third Cantor set) is zero everywhere except in the singularity points where it is finite.

In the following, the local fractional operators will be applied to extend the continuum mechanics equations to fractal media. Fractional static and kinematic equations will be obtained and the Principle of Virtual Work will be demonstrated for fractal media.

5. Kinematic and static equations for fractal media

5.1. Kinematic equations

Let us consider an homeomorphic displacement field $\{\eta\}$. If the hypothesis of small deformations (small displacement gradient $\nabla\{\eta\}$) is adopted, the Cauchy–Green strain tensors reduce to the infinitesimal strain tensor $[\varepsilon]$ given by

$$[\varepsilon] = \frac{1}{2}\left(\nabla\{\eta\} + \nabla\{\eta\}^{\mathrm{T}}\right). \tag{17}$$

Due to the by-definition symmetry of this second-order tensor, the above kinematic equation can be written in the operatorial notation by substituting $[\varepsilon]$ with the infinitesimal strain vector $\{\varepsilon\}^{\mathrm{T}} = (\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})$. In a Cartesian frame of reference (X, Y, Z), the displacement vector is $\{\eta\} = (u, v, w)$ and thus

$$\{\varepsilon\} = [\partial]\{\eta\}, \tag{18}$$

where $[\partial]$ is the differential matrix operator (6×3) containing the first partial derivatives with respect to the three independent variables (x, y, z). It naturally comes out that classical strain represents a nondimensional field.

When a fractal body is considered, the displacement field maintains the dimension of length. Instead, strain is no longer nondimensional and, as shown in Section 2.3, it assumes noninteger dimensions: $[\varepsilon^*] = [\mathrm{L}]^{d_\varepsilon}$. Therefore, it can be obtained by fractional differentiation of the displacement vector, according to the definition of LFDs outlined in the previous section. The fractional differential operator $[\partial^\alpha]$ can thus be introduced, where the order of differentiation is $\alpha = 1 - d_\varepsilon$. Thereby, the kinematic equations for the fractal medium become

$$\{\varepsilon^*\} = [\partial^\alpha]\{\eta\}, \tag{19}$$

and are valid at any point of the body. Classical strain is obtained when $\alpha = 1$. Instead, when $\alpha = 0$, strain is no longer homogeneously diffused inside the medium, but reduces to localized displacement discontinuities. The intermediate situations are described by generic values of α.

5.2. Static equations

In continuum mechanics, body forces $\{\mathscr{F}\} = (\mathscr{F}_x, \mathscr{F}_y, \mathscr{F}_z)$ are considered, acting at any point of the medium, with the physical dimensions of $[\mathrm{F}][\mathrm{L}]^{-3}$. According to the elementary equilibrium equations, these forces balance the divergence of the stress tensor $[\sigma]$ at any point. If an operatorial notation is chosen, the six-components stress vector $\{\sigma\}^{\mathrm{T}} = (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})$ can be adopted in place of the second-order stress tensor $[\sigma]$, due to the symmetry of the tensor. Therefore, the equations of equilibrium can be written as

$$[\partial]^{\mathrm{T}}\{\sigma\} = -\{\mathscr{F}\}, \tag{20}$$

where $[\partial]^{\mathrm{T}}$ is the transposed of the kinematic differential operator $[\partial]$.

In addition, equivalence at the boundary of the body requires that the stress vector coincides, at any point of the boundary, with the applied surface force $\{p\} = (p_x, p_y, p_z)$

$$[N]^{\mathrm{T}}\{\sigma\} = \{p\}, \tag{21}$$

where $[N]^{\mathrm{T}}$ is the cosine matrix of the outward normal vector on the boundary. A perfect correspondence exists between the operatorial matrix $[\partial]^{\mathrm{T}}$ and the geometrical matrix $[N]^{\mathrm{T}}$, i.e., the partial derivatives in the former correspond to the related cosines in the latter.

In the case of fractal media, body forces assume noninteger dimensions according to the Hausdorff dimension Δ_ω of the deformable subset, i.e., $\{\mathscr{F}^*\} = [\mathrm{F}][\mathrm{L}]^{-(3-d_\omega)}$. On the other hand, as shown in Section 2.2, the fractal stress tensor $[\sigma^*]$ presents anomalous dimensions $[\mathrm{F}][\mathrm{L}]^{-(2-d\sigma)}$. Therefore, the equations of equilibrium for the fractal body can be written as

$$[\partial^\alpha]^{\mathrm{T}}\{\sigma^*\} = -\{\mathscr{F}^*\}, \tag{22}$$

where the static fractional differential operator $[\partial^\alpha]^{\mathrm{T}}$ is the transposed of the kinematic fractional differential operator $[\partial^\alpha]$. It is worth to observe that the fractional order of differentiation of the static operator in the fractal medium is $\alpha = 1 - d_\varepsilon$, the same of the kinematic operator (Eq. (19)). This remarkable result is due to the fundamental relation among the exponents (Eq. (10)), and represents the *Duality Principle for Fractal Media*.

Equivalence at the boundary of the body requires that the stress vector defined by Eq. (5) coincides with the applied fractal boundary forces $\{p^*\}$ (with physical dimensions $[\mathrm{F}][\mathrm{L}]^{-(2-d_\sigma)}$)

$$[N]^{\mathrm{T}}\{\sigma^*\} = \{p^*\}. \tag{23}$$

Note that, in the case of fractal bodies, $[N]^{\mathrm{T}}$ is defined, *at any dense point of the boundary*, as the cosine matrix of the outward normal to the boundary of the initiator of the fractal body (see, for instance, Fig. 1).

6. Principle of Virtual Work for fractal media

The Principle of Virtual Work is the fundamental identity of solid mechanics. It affirms the equality between the *virtual external work* (done by body forces and boundary tractions) and the *virtual internal work* (done by internal stresses). More precisely, the principle represents itself the definition of the internal work of deformation, as the scalar product of the stresses times the strains, and affirms that, given any statically admissible system of body forces $\{\mathscr{F}_A\}$ and stresses $\{\sigma_A\}$ acting in the volume V, and boundary tractions $\{p_A\}$ acting upon the smooth boundary S, the internal work balances the external work for any independent kinematically admissible system of displacements $\{\eta_B\}$ and strains $\{\varepsilon_B\}$ in V

$$\int_V \{\mathscr{F}_A\}^{\mathrm{T}}\{\eta_B\}\,\mathrm{d}V + \int_S \{p_A\}^{\mathrm{T}}\{\eta_B\}\,\mathrm{d}S = \int_V \{\sigma_A\}^{\mathrm{T}}\{\varepsilon_B\}\,\mathrm{d}V. \tag{24}$$

As is well known, the proof of the principle requires the application of the Gauss–Green theorem. Some attempts of extending the Gauss–Green theorem to fractal domains have been made in the past. In the framework of classical fractional calculus, it is worth to mention the fractional generalization of the integration by parts obtained by Love and Young [34] using the Riemann–Liouville definitions. No step forward has been made since then. Unfortunately, since only fractional derivatives appear while the integrals maintain integer order, the integration by parts obtained by Love and Young [34] cannot provide the right dimensional balance between the physical quantities involved in the theorem.

In the framework of fractal geometry, Panagiotopoulos and Panagouli [4] made use of the peculiar properties of the Besov spaces to write the Gauss–Green theorem for an Euclidean medium with fractal boundaries. Another approach was adopted by Harrison and Norton [35], but their conclusions are still restricted to compact bodies with fractal boundaries. In all these approaches, fractality is limited to the boundary of the body, and no connection between fractal geometry and fractional calculus was traced. This limitation is overcome by the definition of local fractional operators (Eqs. (15) and (16)). In fact, following Kolwankar and Gangal [32,33], the formula of *local* fractional integration by parts can be written, in one dimension, as (see Appendix A):

$$[f(x)g(x)]_a^b = \{I^\alpha[g(x)D^\alpha f(x)]\}_a^b + \{I^\alpha[f(x)D^\alpha g(x)]\}_a^b, \tag{25}$$

where both integrals and derivatives are fractional [12]. Therefore, dimensional homogeneity is guaranteed by Eq. (25). Note, however, that the above formula is restricted to functions possessing the same critical order of derivation α, and that the order of integration coincides with the order of derivation.

Consider now two arbitrary spatial functions $f(x,y,z)$ and $g(x,y,z)$, defined in a fractal domain Ω^*, with the same critical order α. The general formula of local fractional integration by parts can be obtained by taking the fractal integral of order $\beta-\alpha$ (with $\beta>\alpha$) of both sides of Eq. (25)

$$\left\{I^{(\beta-\alpha)}[gf]\right\}_{\Gamma^*} = \left\{I^\beta[gD^\alpha f]\right\}_{\Omega^*} + \left\{I^\beta[fD^\alpha g]\right\}_{\Omega^*}, \tag{26}$$

where Γ^* is the boundary of the domain Ω^*. The above equation represents the generalization of the Gauss–Green theorem to fractal domains in 3D. It has been obtained by means of the local fractional operators. Note that also the boundary integral is fractional, and its order must be equal to $\beta-\alpha$.

Based on Eq. (26), the Principle of Virtual Work for fractal media can be easily demonstrated. The work of the fractal body forces can be written as

$$L_{\Gamma^*} = \int_{\Omega^*(3-d_\omega)} \{\mathscr{F}_A^*\}^{\mathrm{T}}\{\eta_B\}\,\mathrm{d}\Omega^* = -\int_{\Omega^*(3-d_\omega)} \left(\left[\partial^{(1-d_\varepsilon)}\right]^{\mathrm{T}}\{\sigma_A^*\}\right)^{\mathrm{T}}\{\eta_B\}\,\mathrm{d}\Omega^*, \tag{27}$$

where Ω^* is the fractal domain (with dimension $3-d_\omega$) and the fractal static equation (22) has been applied to the last term. Note that, in order to clarify the mathematical procedure, the subscript of the integral represents the integration domain with the (fractional) order of integration into brackets. Applying to the last member of Eq. (27) the fractal Gauss–Green theorem (26), with $\alpha = 1-d_\varepsilon$ and $\beta = 3-d_\omega$, one obtains

$$L_{\Gamma^*} = \int_{\Omega^*(3-d_\omega)} \{\sigma_A^*\}^{\mathrm{T}} \left[\partial^{(1-d_\varepsilon)}\right] \{\eta_B\} \, \mathrm{d}\Omega^* - \int_{\Gamma^*(2-d_\omega+d_\varepsilon)} \{\sigma_A^*\}^{\mathrm{T}} [N] \{\eta_B\} \, \mathrm{d}\Gamma^*, \tag{28}$$

where Γ^* is the boundary of the domain Ω^* and $[N]^{\mathrm{T}}$ is the cosine matrix of the outward normal vector, defined as the normal to the boundary of the initiator of the fractal body. Applying the kinematic equation (19) and the fundamental relation (10), respectively to the first term and to the integration order of the second term in the right-hand side of Eq. (28), one obtains

$$L_{\Gamma^*} = \int_{\Omega^*(3-d_\omega)} \{\sigma_A^*\}^{\mathrm{T}} \{\varepsilon_B^*\} \, \mathrm{d}\Omega^* - \int_{\Gamma^*(2-d_\sigma)} \{p_A^*\}^{\mathrm{T}} \{\eta_B\} \, \mathrm{d}\Gamma^*, \tag{29}$$

where $\{p_A^*\}$ are the fractal tractions on the boundary Γ^*, as defined by Eq. (23). Comparing Eqs. (27) and (29), the proof is obtained. The Principle of Virtual Work for fractal media can finally be expressed as

$$\int_{\Omega^*(3-d_\omega)} \{\mathscr{F}_A^*\}^{\mathrm{T}} \{\eta_B\} \, \mathrm{d}\Omega^* + \int_{\Gamma^*(2-d_\sigma)} \{p_A^*\}^{\mathrm{T}} \{\eta_B\} \, \mathrm{d}\Gamma^* = \int_{\Omega^*(3-d_\omega)} \{\sigma_A^*\}^{\mathrm{T}} \{\varepsilon_B^*\} \, \mathrm{d}\Omega^*. \tag{30}$$

Note that both sides of Eq. (30) possess the dimensions of work ([F][L]), since the operators are *fractal integrals*, according to Eq. (15), defined upon fractal domains. It can be concluded that Eq. (24), valid for Euclidean media, is a particular case of the more general Eq. (30).

The external work may be done by fractal body forces $\{\mathscr{F}^*\}$ and/or by fractal tractions $\{p^*\}$ acting upon the boundary Γ^* of the sponge. According to the lacunar dimension of the boundary, the anomalous tractions $\{p^*\}$ hold the same noninteger dimensions of internal stresses $[\mathrm{F}][\mathrm{L}]^{-(2-d_\sigma)}$. The internal work is done by the fractal stress $\{\sigma^*\}$ times the fractal strain $\{\varepsilon^*\}$. If an infinitesimal portion of the fractal medium is considered, the unit work of deformation is thus defined as

$$\mathrm{d}W^* = \{\sigma^*\}^{\mathrm{T}} \{\mathrm{d}\varepsilon^*\}, \tag{31}$$

whose dimensions ($[\mathrm{F}][\mathrm{L}]^{-(2-d_\omega)}$) exactly correspond to those of Eq. (3), thanks to the fundamental relation (10) among the exponents.

7. Conclusions

In this paper, the topological framework for the mechanics of deformable fractal media has been outlined. The Menger sponge has been introduced as an archetype of fractal spongy bodies. By means of renormalization group transformations, the fractal stress $[\sigma^*]$ and the fractal strain $[\varepsilon^*]$ have been defined inside this fractal object as the quantities governing the process of deformation. This extends the previous results by the authors [11,18], which were limited only to the corresponding critical quantities. A fundamental relation among the Hausdorff dimension of the deformable subset and the physical dimensions of fractal stress and fractal strain has been obtained.

In the second part of the paper, new mathematical operators from fractional calculus have been applied to write the field equations for solids with fractal lacunar structure. It has been shown that the classical fractional calculus by Riemann–Liouville, due to its nonlocal character, cannot describe properly the deformations of fractal media. Instead, the local fractional operators, recently introduced by Kolwankar [31], can be successfully applied for our purposes. The static and kinematic equations for fractal media have been obtained, and the duality principle of solid mechanics has been shown to hold also in the fractal framework. Finally, the extension of the Gauss–Green theorem to fractal quantities has been proposed, naturally yielding the Principle of Virtual Work for fractal media.

The next step is to define proper constitutive laws (e.g. elastic or softening) for fractal media, and to demonstrate the basic variational principles. Indeed, the dependence of the fractal dimensions of mechanical quantities on the loading level should also be considered. At this stage, only the formal structure of the static and kinematic equations has been outlined. Research is open in this direction, and the practical use of local fractional operators still has to be developed. However, once this goal were achieved, boundary

value problems on fractal sets could be solved, not only in principle, by means of the Local Fractional Calculus.

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Appendix A

Kolwankar and Gangal [32] have shown that a fractal function $f(x)$ (e.g., the Weierstrass function), continuous but nowhere differentiable, is locally fractionally differentiable up to a "critical order" α between 0 and 1. More precisely, the LFD is zero if its order q is lower than α, does not exist if greater than α, and finally exists and is finite only if q is equal to α. Thus, the LFD shows a behaviour analogous to the Hausdorff measure of a fractal set. Furthermore, they have shown that the critical order α is equivalent to the local Hölder exponent of $f(x)$, by proving the following local fractional Taylor expansion for $N < q < N+1$ and $x \to y$:

$$f(x) = \sum_{n=0}^{n=N} \frac{f^{(n)}(y)}{\Gamma(n+1)}(x-y)^n + \frac{D^q f(y)}{\Gamma(q+1)}(x-y)^q + R_q(x-y), \tag{A.1}$$

where R_q is a residual, negligible if compared with the other terms. If $q < 1$, the previous expression becomes

$$f(x) = f(y) + \frac{D^q f(y)}{\Gamma(q+1)}(x-y)^q + R_q(x-y). \tag{A.2}$$

Note that the terms in the right-hand side of Eqs. (A.1) and (A.2) are nontrivial and finite only if q is equal to the critical order α. Moreover, for $q = \alpha$, the fractional Taylor expansion (Eq. (A.2)) gives us the geometrical interpretation of the LFD. When q is set equal to unity, one obtains from (A.2) the equation of a tangent. All the curves passing through the same point y, with the same first derivative, have the same tangent. Analogously, all the curves with the same critical order α (and the same D^α) form an equivalence class modelled by x^α. This is how it is possible to generalize the geometric interpretation of derivatives in terms of "tangents".

Consider now two functions $f(x)$ and $g(x)$ having the same Hölder exponent α at the same point $x = y$. Based on Eq. (A.2), it is easy to prove that:

1. in $x = y$ the product function $h(x)$ has the same Hölder exponent α, unless both the functions $f(x)$ and $g(x)$ attain zero value;
2. in $x = y$ the LFD of order α of the product function can be computed using the classical rule for the differentiation of the product. That is:

$$D^\alpha h(y) = f(y)D^\alpha g(y) + g(y)D^\alpha f(y). \tag{A.3}$$

Finally, consider two continuous functions $f(x)$ and $g(x)$ defined upon $[a,b]$. Suppose now that both these functions have first derivative equal to zero everywhere except at the points belonging to the same fractal set where they possess Hölder exponent α equal to the dimension of their fractal support (i.e., $f(x)$ and $g(x)$ are Cantor staircase-like functions). Hence, Eq. (A.3) is valid on the whole interval $[a,b]$.

Performing, for both sides of Eq. (A.3), the fractal integration of order α upon $[a,b]$ (with $x \in [a,b]$), yields the following formula of fractal integration by parts:

$$[f(x)g(x)]_a^b = \{I^\alpha[g(x)D^\alpha f(x)]\}_a^b + \{I^\alpha[f(x)D^\alpha f(x)]\}_a^b, \tag{A.4}$$

which is coincident with Eq. (25).

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