ANOMALOUS AND IRREGULAR MECHANICAL BEHAVIOUR IN HETEROGENEOUS MATERIALS: SNAP-BACK INSTABILITIES AND FRACTAL CRACKING

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Abstract

Size-scale effects on the ductility (or brittleness) of structural behaviour are considered and explained in the framework of *Dimensional Analysis*, by applying Linear Elastic Fracture Mechanics and Plastic Limit Analysis, or nonlinear crack models including the effects of the plastic or cohesive process zone ahead of the stress-free crack tip.

When the size-scale effects regard the strength and toughness parameters themselves, which are generally and traditionally considered as scale-invariants, a physically-based explanation may be proposed within the wider conceptual framework of *Fractal Geometry* and *Renormalization Group Theory*. The systematic tensile strength decrease and fracture energy increase with size are respectively connected with the noninteger physical dimensions of material cross-sections and fracture surfaces. Whereas the former are lower than 2, due to self-similar voids and cracks, the latter are higher than 2, due to multiscale tortuosity and complex roughness.

1. Introduction

A very clear trend in structural mechanics is that of the larger elements which behave in a brittle way and fail under relatively low stresses. They appear to be particularly brittle and weak, as well as the smaller elements appear to be particularly ductile and strong. Two historical examples are respectively provided by the Liberty Ships and by the glass filaments (*whiskers*). Whereas the latter gave inspiration to Griffith (1921), so the former pushed Irwin (1957) and other scientists to found modern fracture mechanics.

Even at the scale of the laboratory the same trends emerge as those described previously for extreme cases. Larger specimens fail suddenly with a crack propagating in an unstable manner through an elastic and undamaged material. On the contrary, smaller specimens fail slowly with a crack propagating in a stable manner through a ductile and damaged material.

A first explanation to such unexpected and surprising phenomena was given by the well-known Griffith's formula: $\sigma = K_{IC}\sqrt{\pi a}$ where σ is the reduced strength, K_{IC} is the fracture toughness of the material and *a* is the half-length of the pre-existing crack or, more generally, a scale-parameter. On the other hand, while the scale effects on the *brittleness* (or *ductility*) of the structural behaviour can be explained and interpreted in the classical framework of Dimensional Analysis, considering the different physical dimensions of *tensile strength* and *fracture energy*, the scale effects on the last two material properties, generally and traditionally considered as scale-invariants, can be explained and interpreted only in a non-classical and wider conceptual framework, leading to *Fractal Geometry* and to *Renormalization Group Theory*.

2. Size Effects on Structural Brittleness

2.1. Cohesive crack model

The cohesive crack model is based on a double constitutive law: stress vs. strain (ascending branch) and stress vs. crack opening displacement (descending branch) (Fig. 1.a,b). Consistently, the cracked zone in a disordered material is idealized as formed by two parts: (1) stress-free crack, and (2) process zone, where the softening stresses are transmitted (Fig. 1.c,d).

Mortar presents a fracture energy G_F which is lower than that of concrete. Due to a considerable decrease of G_F, the mechanical response appears to be much brittler and the snap-back instability provides unstable branches with positive slope. When the macrocrack grows, thus the stresses relax and the strains decrease, so that a global decrease in both load and deflection occurs.

On the other hand, it is nowadays well-known that the same trend is also provided by increasing the size-scale b of the specimen or by increasing the tensile strength σ_u of the material. If we group the three fundamental parameters in the *energy brittleness number* (Carpinteri, 1985, 1989a, 1989b):

$$s_E = \frac{G_F}{\sigma_u b}, \tag{1}$$

it is simple to represent the load vs. deflection diagrams in a non-dimensional manner and to emphasize the ductile-brittle transition (Fig. 2.a).

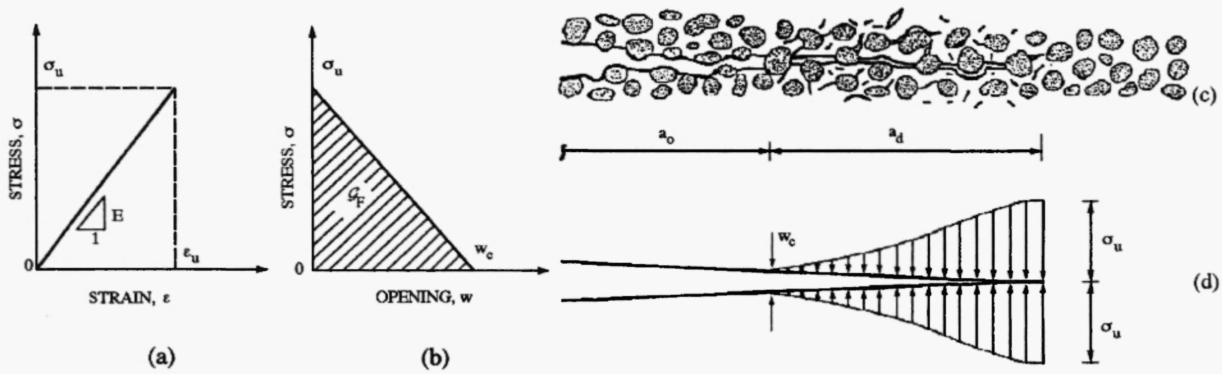

Figure 1: Double constitutive law: (a) stress vs. strain and (b) stress vs. crack opening displacement; (c) process zone in front of the stress-free crack and (d) cohesive crack model.

In the case of an initially uncracked specimen, the apparent flexural strength decreases by increasing the beam depth b, from the value 3 σ_u to σ_u (Fig. 3.a). This is due to the collapse mechanism, which presents an ultimate moment M_{max} that is exactly three times that of an elastic-perfectly brittle material and twice that of an elastic-perfectly plastic material.

In the case of an initially cracked specimen, the apparent fracture toughness increases by increasing the beam depth b, from zero to the true value K_{IC} (Fig.3.b). As a matter of fact, when the *static brittleness number* (Carpinteri,1981a, 1982):

$$s = \frac{K_{IC}}{\sigma_u b^{1/2}}, \tag{2}$$

tends to zero (larger specimens), the LEFM instability tends to prevail over the plastic collapse at the ligament.

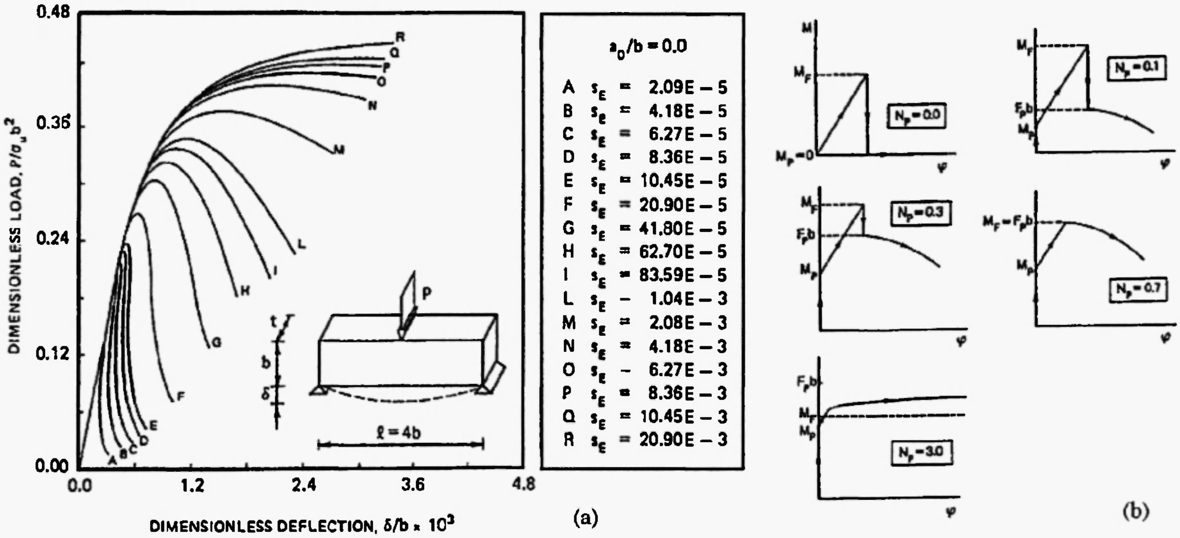

Figure 2: (a) Ductile-brittle transition by varying the brittleness number $s_E = G_F/\sigma_u b$; (b) bending moment vs. local rotation diagrams by varying the brittleness number $N_P = (f_y b^{1/2}/K_{IC})(A_S/A)$.

2.2. Bridged crack model

The bridged crack model with only one fibre can be considered as a redundant mechanical system, where the unknown force F transmitted by the reinforcement to the beam is obtained from a rotational compatibility condition (Carpinteri, 1981b, 1984).

The stability of the combined process of concrete fracture and steel plastic flow (or slippage) was analyzed by the first author by applying the superposition principle: $K_I = K_I(M) - K_I(F) = K_{IC}$, where the terms $K_I(M)$ and $K_I(F)$ are the stress-intensity factors due respectively to bending moment and to reinforcement action. The moment vs. rotation diagrams are reported in Fig. 2.b, by varying the *composite brittleness number* (Carpinteri, 1981b, 1984):

$$N_p = \frac{f_y b^{1/2}}{K_{IC}} \frac{A_s}{A} \tag{3}$$

where f_y is the yield (or slippage) strength of steel, K_{IC} is the fracture toughness of concrete and A_S/A is the steel percentage. The diagrams present a negative jump, which vanishes for $N_p \geq 0.7$. The condition $N_p \approx 0.7$ represents an optimum situation of minimum reinforcement without material instabilities. This was found very useful in the case of high strength concrete, to which LEFM is applicable (Bosco et al., 1990; Bosco and Carpinteri, 1992).

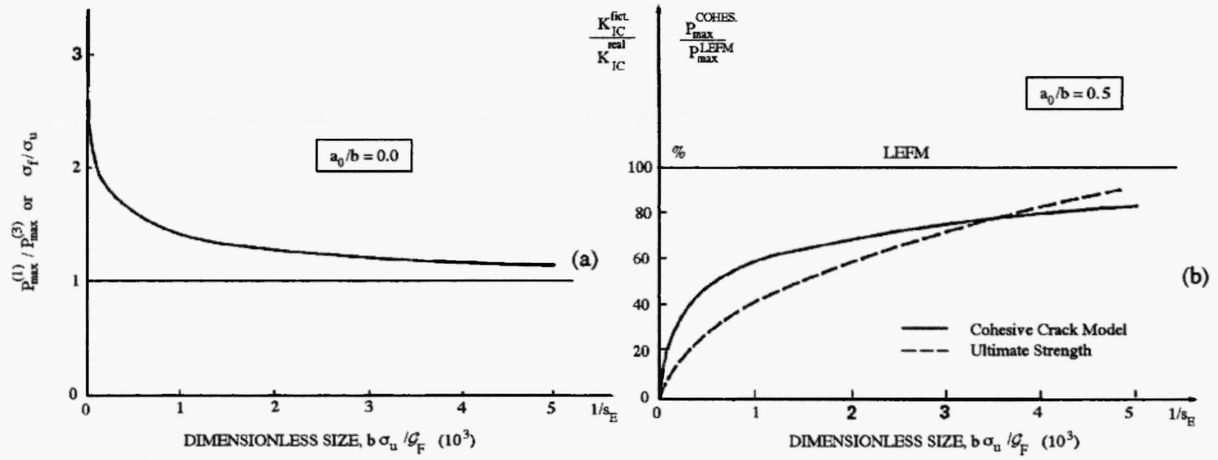

Figure 3: (a) Flexural tensile strength decrease with specimen size and (b) fictitious fracture toughness increase with specimen size.

3. Size Effects on Material Strength and Toughness

3.1. Lacunar fractality of material ligaments

Lacunar fractal sets present a dimension lower than that of reference (Mandelbrot, 1982). The archetype is represented by the Cantor dust (or middle third set). It may be constructed from a unit interval by a sequence of deletion operations (Fig. 4.a). Its classical length is zero, whereas it is possible to demonstrate that its physical dimension is $[L]^{0.631}$.

In the two-dimensional space, the lacunar fractals present an intermediate dimension between one and two. They are very rarefied areas with holes at any scale. A typical example was given by Sierpinski (Fig. 4.b).

In the case of a plane cross-section with holes, pores, cracks or craters, the resisting portion can be considered as a lacunar fractal. It is possible to prove that, at the peak tensile stress, the damaged cross-section presents the dimension 1.5 (Carpinteri, 1994a, 1994b).

In the three-dimensional space, the lacunar fractals present an intermediate dimension between two and three. The so called Menger sponge (Fig. 4.c), which is nothing but a generalization of the Cantor dust and of the Sierpinski carpet, presents the dimension 2.73. It is an archetype for porous solids. The density of such solids is not uniform neither constant. If considered as a mass divided by a volume, it would decrease with the solid size. The density could become a material scale-invariant constant only dividing the mass by the lacunar volume, which presents an anomalous dimension lower than three. On the other hand, if we consider the resisting cross-section of a solid in tension at the peak stress as a lacunar fractal of dimension 1.5, we can justify the nominal tensile strength decrease as the inverse of the square root of the specimen size. Therefore, a situation completely analogous to that of the sponge density occurs. Even the strength, as the density, decreases with the scale. In order to obtain a scale-invariant quantity, it is necessary to divide the maximum force by the effectively reacting lacunar area, which presents the anomalous dimension 1.5. The pseudo-strength obtained in this way is not a significant parameter from the engineering view-point, since its anomalous physical dimension,

$[F][L]^{-3/2}$, is rather unusual. Such procedure is called *renormalization* and has been previously proposed in Statistical Physics, Field Theory and Turbulence.

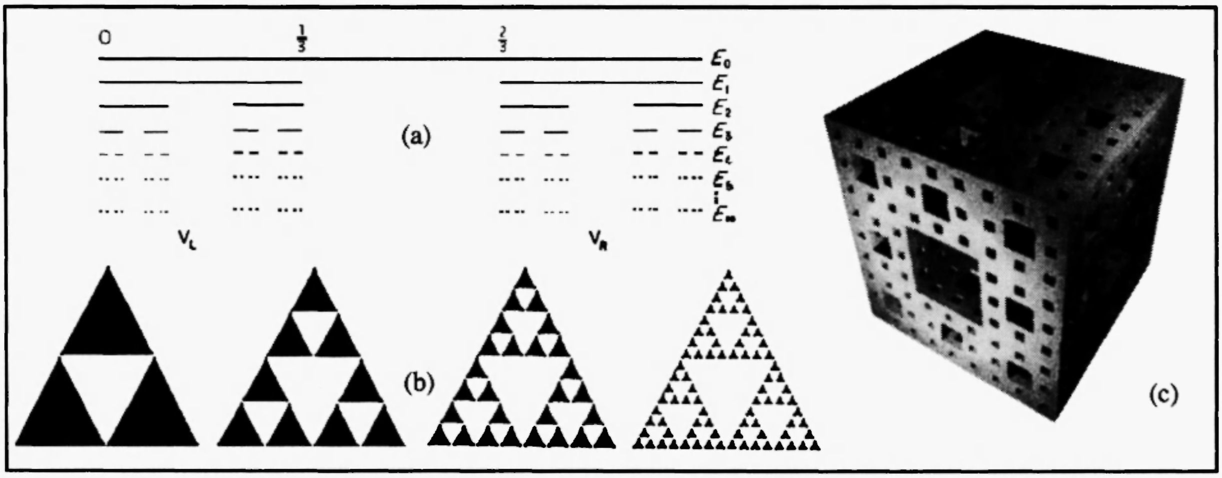

Figure 4: (a) Middle-third Cantor set (fractal dimension = 0.631); (b) Sierpinski gasket (fractal dimension = 1.58) and (c) Menger sponge (fractal dimension = 2.73).

3.2. Invasive fractality of fracture surfaces

Invasive fractal sets present a dimension higher than that of reference (Mandelbrot, 1982). The archetype is represented by the von Koch curve (Fig. 5.a). It may be constructed by a sequence of replacements of middle thirds by the other two sides of the equilateral triangle based on the removed segment. Its classical length is infinite, whereas it is possible to demonstrate that its physical dimension is $[L]^{1.262}$.

When the generator is particularly tortuous, as that in Fig. 5.b, already at the first iteration the shape appears very complex, and, in any case, not suitable to simulate crack trajectories but, rather, microcrack distributions.

In the three-dimensional space, the invasive fractals present an intermediate dimension between two and three. They are extremely rough surfaces, with peaks and valleys at any scale. If we exclude particularly tortuous protuberances, as those in Fig. 5.b, the maximum dimension reachable by the invasive fractal surfaces is 2.5 (Brownian surfaces).

On the other hand, it is experimentally observed that the fracture energy increases with the specimen size. If we consider the final fracture surface as an invasive fractal of dimension 2.5, it is possible to justify such an increase. The scaling effect is positive in this case.

In order to obtain a scale-invariant property, it is necessary to divide the dissipated energy by the invasive area of dimension 2.5. The renormalized fracture energy presents an anomalous physical dimension, $[F][L]/[L]^{5/2}=[F][L]^{-3/2}$, which curiously is the same as that of the renormalized tensile strength.

In the characterization of building materials, relatively small specimens are used with respect to the size of the structural elements. This represents a sensible perturbation of the real properties, providing overestimated strengths as well as underestimated toughnesses. In the case of microscopical specimens, the tensile strength can appear higher than the usual one even by orders of magnitude, as well as the fracture energy can appear as much lower than the conventional one (Carpinteri and Ferro, 1994; Carpinteri and Chiaia, 1996).

To these tendencies various phenomenological explanations have been proposed, through physical and analytical modelling, which, sometimes, have turned to appear rather simplistic. Such interpretations may be based on *Dimensional Analysis* only when flexural stresses prevail (Fig. 3.a), or when an initial crack or notch is present in the specimen (Fig. 3.b). In the case of tensile testing without initial notches, *Fractal Geometry*, with the physical procedures of renormalization, appears to provide a more generalized and unified framework.

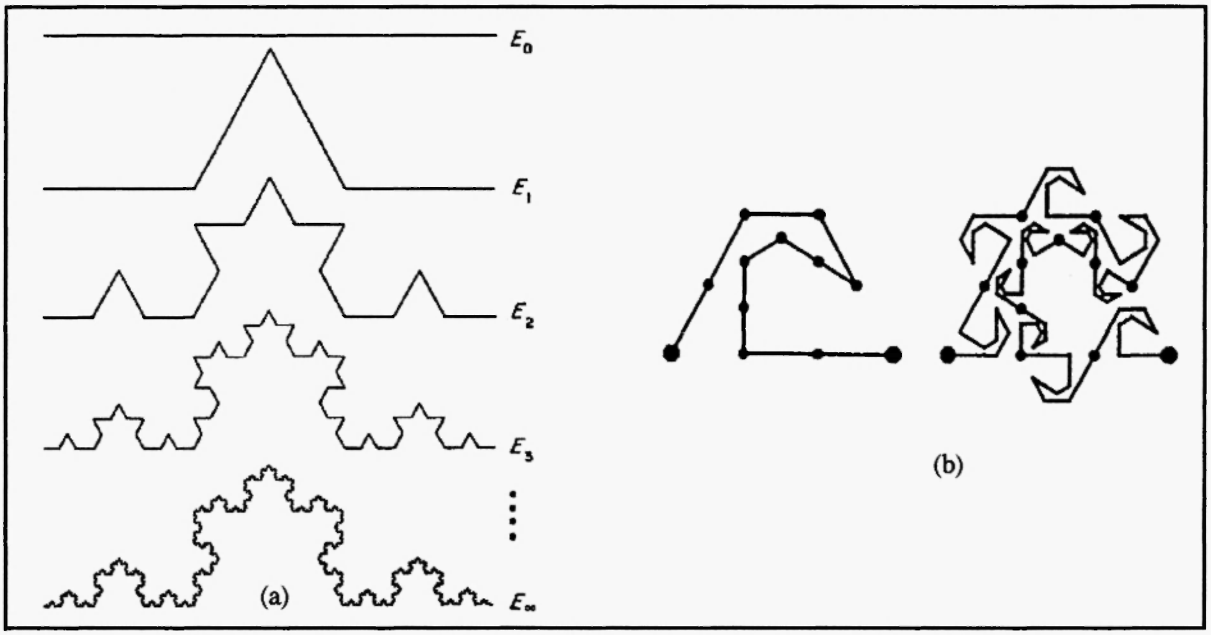

Figure 5: (a) Von Koch curve (fractal dimension = 1.262) and (b) particularly tortuous generator providing a very intricate fractal curve (fractal dimension = 1.87).

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