Size Dependence of Strength and Fracture Properties of Brick Masonry Walls

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Abstract: The problem of size dependence of the fracture properties in brick masonry structures is addressed. The constitutive heterogeneity of the material seems to be responsible for this effect, which provides multiscale cooperation as fracture develops. Three-point bending tests have been carried out on notched masonry walls with five different sizes, and the nominal ultimate strength and fracture energies have been computed according to conventional theories. It is shown that the multifractal hypothesis for the composite mesostructure of the masonry allows for the proper determination of the scaling regimes of these physical quantities. By means of two multifractal scaling laws, for strength and toughness, respectively, the asymptotic values of these quantities, valid for real-sized structures, and the threshold scale between the disordered regime and the homogeneous one can be determined.

INTRODUCTION: HIERARCHICAL STRUCTURE OF BRICK MASONRY WALLS

Brick masonry can be considered as a composite material in a general sense, since it consists of two different phases that form regions large enough to be regarded as continua and which are usually firmly bonded together at the interface. On the other hand, brick masonry represents a peculiar kind of composite, being neither a particulate material nor a fibrous one. In fact, when one phase is in the form of particles embedded in a matrix phase, the material is called a particulate composite (Hashin 1983), which is the case of plain concrete (macroscopically isotropic material). If the particles can be considered as (aligned) monodimensional cylinders, a fiber composite is obtained, which is highly anisotropic compared with particulate composites.

In the case of brick masonry, the two constitutive phases, i.e., the bricks and the mortar beds, are assembled in such a way that the volume of the first phase is larger than that of the second phase. This provides orthotropic mechanical behavior of the composite. On the other hand, the global mechanical behavior of the masonry is strongly controlled by the mortar beds, that is, by their thickness and their strength and deformation properties, and cannot be easily interpreted by means of the classical structural mechanics models (Hendry 1981).

The statistical analysis of the mechanical properties of composites originated from a famous paper by Einstein (1906), in which he computed the effective viscosity of a fluid containing a small amount of rigid spherical particles. The research has since then been primarily concerned with macroscopically isotropic composites (matrix/particle composites and polycrystalline aggregates), and the results have been simply extended to the case of brick masonry. In most of the proposed models a mean-field approach can be recognized where a "representative volume element" (RVE) is defined, which is large compared with the typical phase region dimensions (see Fig. 1), and whose mechanical properties are the same in any position of the body ("statistical homogeneity" of the composite). The problem is then reduced to the determination of the effective properties of the RVE, and this is usually pursued by means of averages of the field variables, such as stress and strain, when their space variation is statistically homogeneous (Christensen 1979).

It is always assumed that such continua retain their properties regardless of the specimen size; thus also for infinitesimal or very large elements. This permits establishment of the field equations in terms of field variable derivatives, resulting in an approximate macrodescription of the composite, which is valid only if the RVE size is "sufficiently larger" than the phase's characteristic dimensions and "sufficiently smaller" when compared with the body dimensions (Anthoine 1995). The aforementioned approach, although adequate for most engineering problems, turned out to be insufficient in the presence of dynamic loadings (e.g., the earthquake resistance of masonry buildings) and of the consequent very high stress and strain gradients (e.g., at the tip of developing cracks). Moreover, the scale dependence of the physical properties is disregarded, yielding poor predictions of the mechanical behavior of real-sized masonry structures.

As it has been realized in many fields of physics (Wilson 1971), the scaling behavior of the physical quantities has to be taken into account in correspondence of critical phenomena. Moreover, the cooperative aspects of the fracture phenomenon in heterogeneous materials claim a different topological description of the considered domains (Carpinteri 1994a). The fracture process in brick masonry structures appears definitely hierarchical in the sense that cracks usually originate from pre-

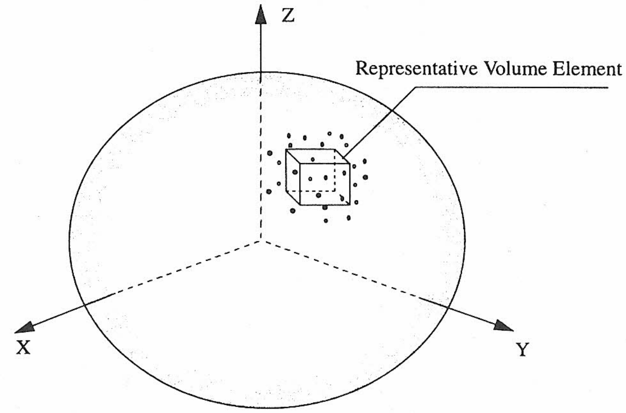

FIG. 1. Representative Volume Element

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existing microdefects in the mortar or at the mortar/brick interface (microlevel). Then they develop at the scale of the mortar beds thickness, involving debonding or even breaking of the closest bricks (mesolevel). Finally they extend to the structural scale (macrolevel), resulting in the typically disordered (zigzag) pattern with multiple cracking randomly diffused at the interfaces, through the bed thickness and through the bricks, as shown in Fig. 2.

Therefore, the degree of correlation in the disordered fracture process has to be taken into proper account, since it is evidently related to the size of the considered specimens and is responsible for the scaling of the mechanical properties.

MULTIFRACTAL SCALING LAWS FOR STRENGTH AND TOUGHNESS

The simplest way to handle the hierarchical microstructural heterogeneity is to define the rules relating to the various scales of observation. When a scale-independent morphology is encountered, the object is called "self-similar." The self-similarity of many disordered domains has been extensively detected (Mandelbrot 1982), allowing for the synthetic description of the considered topologies in terms of fractal dimensions. The fractality of the fracture domains has been revealed by Carpinteri and Chiaia (1995) on concrete fracture surfaces, and it has been detected in many other materials and in a broad range of scales (Davidson 1989). "Invasive" (or densifying) fractal domains, like the von Koch curve, are defined when the topological dimension strictly exceeds the Euclidean one, whereas "lacunar" (rarefying) domains, like the Cantor set, are characterized by dimensions smaller than that of the embedding space.

On the other hand, a unique value of the fractal dimension cannot be defined for natural random objects like masonry, but the multifractal scaling of the physical quantities defined over these domains must be considered. In fact, due to the presence of an upper and a lower bound in the scaling range, the inevitable transition from the fractal (disordered) regime at the smallest scales towards a Euclidean (homogeneous) regime at the largest scales is provided. The upper bound is represented by the macroscopic size b of the structure, while the lower one is related to the size of the smallest phase that comes into the definition of the composite material, which could be, in the case of masonry, the bed thickness.

Bounding the observation range in this way does not exclude different scaling regimes at smaller scales. The bricks and the mortar beds can be considered as different materials themselves, up to the scale of the clay grains and to the scale of the cement paste. A minimum scale is thus implicitly defined (mesoscale), below which brick masonry ceases to be considered a well-defined material. An internal characteristic length l_{ch}, typical of each heterogeneous material, comes into play, which inhibits the development of a perfect self-similar scaling through the whole scale range, whereas mathematical fractals, absolutely lacking of any characteristic length, exhibit uniform (monofractal) scaling without any bound.

Therefore, Carpinteri and Chiaia (1995) defined this kind of scaling as "geometrical multifractality," since an infinity of exponents is necessary in order to describe the entire range of the scaling (Fig. 3). From the mechanical point of view, this anomalous scaling essentially implies that the effect of the mesoscopic heterogeneity on the mechanical behavior of brick masonry becomes progressively less important for larger walls (i.e., large when compared with the characteristic size l_{ch}), where the composite mesostructure is somehow homogenized (Carpinteri 1994b). Therefore, scaling should vanish in the limit of structural size tending to infinity, where asymptotic constant values of the mechanical quantities can be determined and the classical mean-field approach becomes applicable. On the other hand, the highest possible disorder in the masonry fracture patterns seems to be represented by a Brownian disorder at the smallest scales, which proves the $\pm 1/2$ fractal exponent in the scaling laws. Such a thermodynamical assumption (Brownian motion is the archetype of dissipative phenomena) is confirmed by the multitude of fractal dimension measurements in different materials that have been reported in the literature and by the suggestive hypothesis that Brownian patterns would globally minimize the energy dissipation (Chudnowski and Kunin 1987).

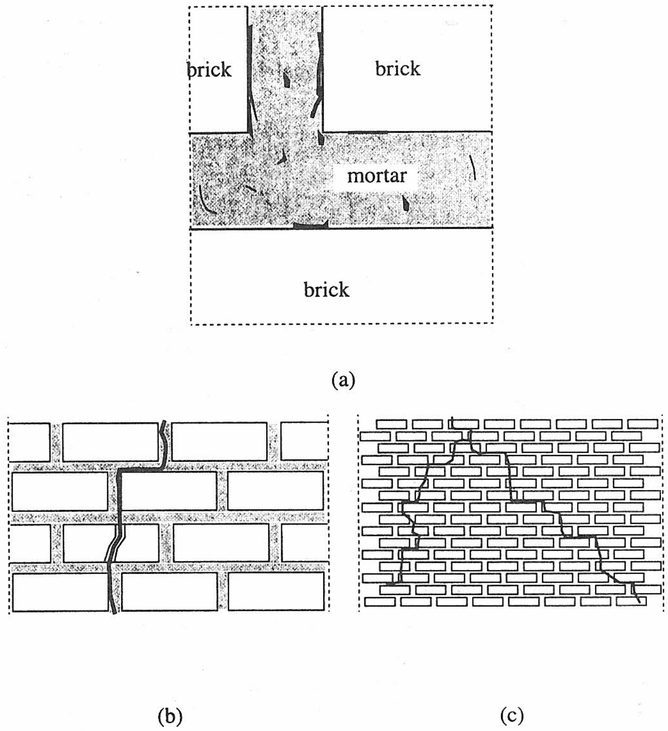


FIG. 2. Multiscale Fracture Process in Brick Masonry: (a) Microscopic Fracture; (b) Mesoscopic Fracture; (c) Macroscopic Fracture

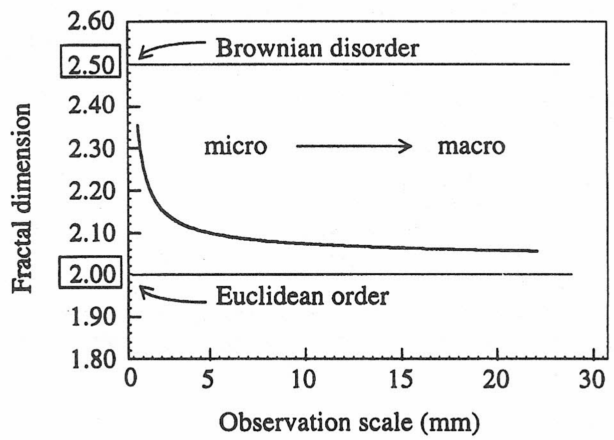


FIG. 3. Geometrical Multifractality of Fracture Patterns

According to the aforementioned arguments, two multifractal scaling laws (MFSLs) have been proposed by Carpinteri and Chiaia (1996) and by Carpinteri et al. (1995) for the nominal fracture energy $\mathcal{G}_F$ and the nominal tensile strength σ_u, which are related to an invasive and to a lacunar fractal domain, respectively. The nominal fracture energy $\mathcal{G}_F$ can be defined as the specific energy dissipated on the (smooth) area of the resisting ligament. It is thus given by the ratio between the total work of fracture and the projected area of the fracture surface. The nominal strength σ_u is a reference parameter usually chosen according to the linear elastic solutions. In the case of three-point bending, it is given by (2). The scaling laws can be written in the following analytical form:

$$\mathcal{G}_F(b) = \mathcal{G}_F^{\infty}\left(1 + \frac{l_{ch}}{b}\right)^{-1/2} \quad (1a)$$

$$\sigma_u(b) = f_t\left(1 + \frac{l_{ch}}{b}\right)^{1/2} \quad (1b)$$

where b is a structural reference size. These scaling laws, shown in Fig. 4, are both two-parameter models, where the asymptotical value of the nominal quantity ($\mathcal{G}_F^{\infty}$ or f_t), corresponding, respectively, to the highest nominal fracture energy and to the lowest tensile strength, is reached only in the limit of infinite sizes. The dimensionless term in square brackets, which is controlled by the characteristic value of the masonry internal length l_{ch}, represents the variable influence of disorder on the mechanical behavior, thus quantifying the difference between the nominal quantity measured at scale b and the asymptotic constant value.

In the bilogarithmic diagrams shown in Fig. 5, the transition from the fractal regime to the Euclidean one becomes evident. The threshold of this transition is significantly represented by point Q, whose abscissa is equal to ln l_{ch}. Note that the oblique asymptotic is controlled in both cases by a quantity with the dimensions of a stress-intensity factor ($[F][L]^{-3/2}$) and by a slope equal to $\pm 1/2$. In the case of fracture energy, in fact, the ratio between the asymptotic toughness $\mathcal{G}_F^{\infty}$ and the square root of the internal length l_{ch} is present, whereas in the case of the strength, the quantity $f_t\sqrt{l_{ch}}$ comes into play (see Fig. 5). This implies that the Griffith collapse, governed by the linear elastic fracture mechanism (LEFM) stress-singularity 1/2, comes into play only when the characteristic size a of microdefects is comparable with the external size (highest disorder); whereas, at larger scales, the correlation among the microcracks increases, leading to the homogenization in the limit of infinitely large sizes (ordered regime).

The internal length parameter becomes important when the scaling behavior of two different materials is compared, as shown in Fig. 6 in the case of fracture energy. It can be stated that, in the case of a finer grained material like concrete, the MFSL is shifted to the left with respect to the case of masonry, the value of l_{ch} being much lower for concrete than for brick masonry. Therefore, two specimens of different materials, with the same structural dimension b_1, besides obviously showing two different values of the nominal fracture energy, have to be set in two different scaling regimes. With reference to Fig. 6, the brick masonry specimen behaves accordingly to the fractal regime, whereas the concrete one lies on the asymptotic branch of the MFSL, thus showing a homogeneous macroscopic behavior. Generally speaking, one has to determine for each material the proper range of scales where the effect of the heterogeneity is predominant and, consequently, the minimum structural size beyond which the local toughness and strength fluctuations are macroscopically averaged and scale-independent values of the mechanical properties can be determined.

In the case of concrete structures, best-fitting of relevant experimental results has already confirmed the soundness of this new approach (Carpinteri and Chiaia 1996).

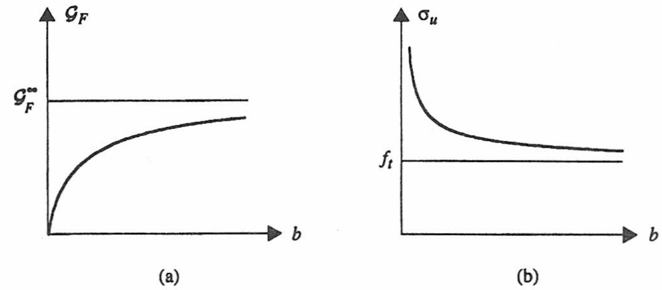


FIG. 4. Multifractal Scaling Laws for: (a) Fracture Energy; and (b) Tensile Strength

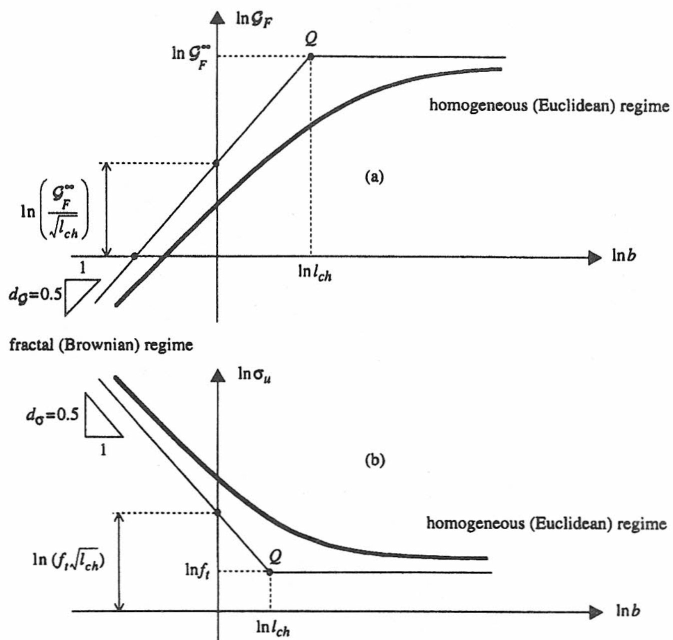


FIG. 5. Multifractal Scaling Laws: Bilogarithmic Diagrams

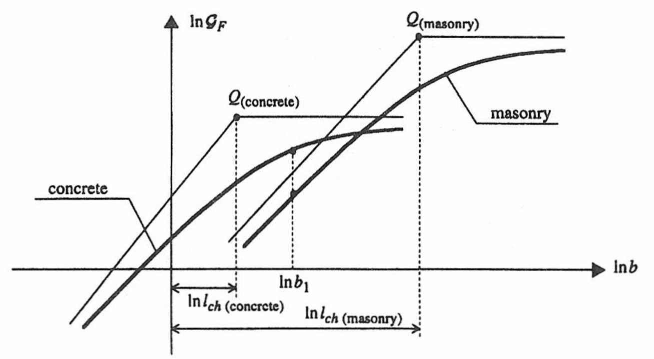
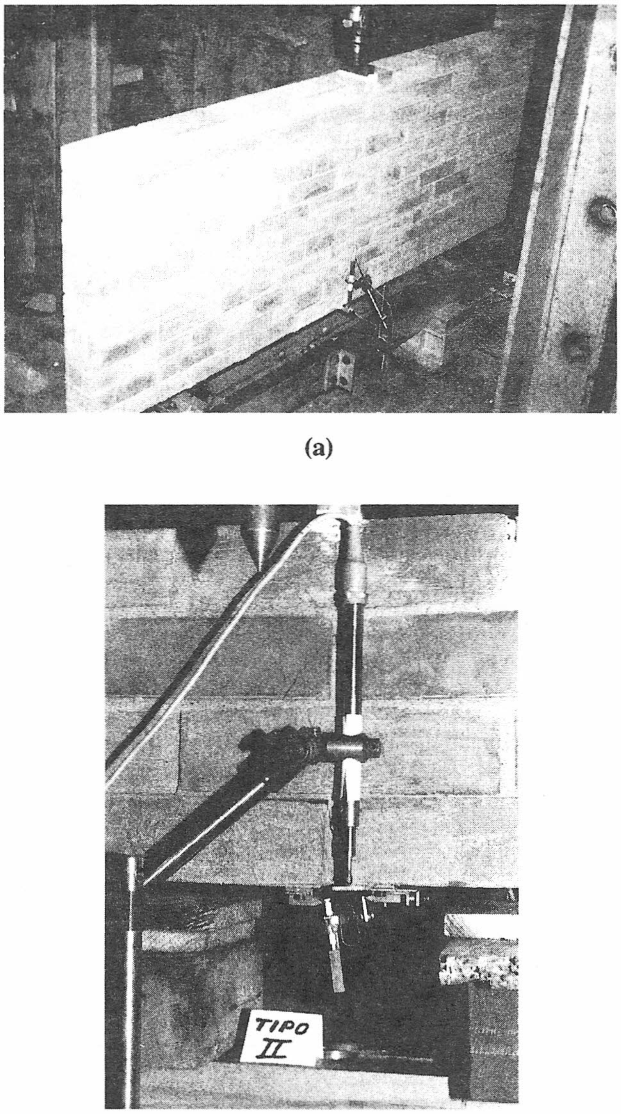


FIG. 6. Different Scaling of Fracture Energy in Masonry and Concrete

EXPERIMENTAL INVESTIGATION: TPB TESTS ON NOTCHED BRICK MASONRY WALLS

The application of nonlinear fracture mechanics concepts to brick masonry structures, although very appropriate from the mechanical point of view, has not been sufficiently investigated up to now. Various attempts to apply the damage mechanics concepts to masonry structures can be found in the literature, but most of them disregard the highly localized character of the breaking process (Marigo 1985) and therefore cannot catch the scale dependence of the involved mechanical quantities. Approaches based on localization limiters can be adopted (De Borst et al. 1993), but the determination of a microstructural parameter related to the internal length is needed. On the other hand, while uniaxial and biaxial compression tests are usually performed on masonry structures (Scott and Abrams 1985), it has become evident that mode I (opening) fracture mechanism is frequent in brick masonry, either in members loaded in static compression or, mainly, in walls subjected to dynamic (seismic) loading (Drysdale and Hamid 1984).

The characterization of the fracture behavior of brick masonry walls can be pursued by means of two physical quantities, the nominal tensile strength σ_u and the fracture energy $\mathcal{G}_F$. While the former is related to the critical point where the transition from the ascending branch to the softening one occurs, the latter takes into account the entire load-displacement behavior and represents the only sound toughness parameter in a highly heterogeneous material like masonry. In fact, the determination of the stress-strain fields at the tips of developing cracks is not needed, and the uncertainty related to the stress-intensity factor is bypassed. On the other hand, both parameters being mean-field quantities, a strong size dependence is provided that can be interpreted by means of the MFSL, as will be demonstrated in the following.



(a)

(b)

FIG. 7. Three-Point Bending Tests on Brick Masonry Walls: Testing Setup

Previous investigations on the clay bricks fracture properties have been described by Bocca et al. (1989), who performed three-point bending tests on notched clay bricks and compared the results to numerical simulations by means of a cohesive crack model originally developed for concrete. These tests on the bricks already demonstrated that fracture mechanics represents a suitable approach to the modeling of masonry structures, especially in order to catch the size-scale transition in their mechanical behavior.

A series of three point bending tests on notched masonry walls has been carried out in the laboratory of the Department of Structural Engineering, Politecnico di Torino (Fig. 7). Five different geometries have been examined, with a minimum of two specimens for each size, resulting in 17 significant tests in the scale range 1:4. It is difficult to achieve a perfect geometrical similarity among the specimens at the smaller scales, due to the fixed dimensions of the bricks. Solid clay bricks have been used, with dimensions 45 × 110 × 250 mm. Bed joints have been realized by means of cementitious mortar, classified as M3 according to Eurocode number 6, with a water/cement ratio equal to 0.6, mixed with hydraulic lime and sand aggregates with maximum size equal to 4 mm. Standard three point bending (TPB) tests have been performed on the clay bricks and on mortar prisms, yielding average values of the fracture energy $\mathcal{G}_F$ of 34.9 and 19.5 N/m, respectively.

The significant geometrical data are summarized in Table 1, where the main results of the TPB tests on the masonry walls are also reported. Displacement-controlled tests have been carried out, by means of a linear variable differential transducer (LVDT), measuring the crack-opening displacement (COD) in correspondence to the notch tip. Note that the notch length a_0 has been set equal to 1/4 of the wall depth d, and the span to depth ratio (b/d) was approximately equal to 2.1. The thickness t of all the walls was set equal to that of the bricks (110 mm). Therefore, the panels were similarly scaled only in the two main directions, while the thickness was kept constant.

The failure mechanism has been in all cases that of a typical bending failure, according to a (macroscopically) Mode I crack propagation. Although, in the early stages of the loading process, an arch-resistant mechanism cannot be excluded, in correspondence of the maximum load (critical point), the bending mechanism prevailed and no shear crack was detected. A reference nominal strength σ_u of the masonry has therefore been computed according to the linear theory of elastic beams, assuming the unnotched ligament ($d - a_0$) as the resisting section:

$$\sigma_u = \frac{6M_u}{t(d - a_0)^2} \qquad (2)$$

where M_u = ultimate moment at midspan. The fracture energy $\mathcal{G}_F$ of the brick masonry walls has been computed as the area

TABLE 1. Three-Point Bending Tests on Brick Masonry Walls

Type of wall (1)	Number of tested specimens (2)	Specimen dimensions (mm) $L \times d \times t$ (3)	Notch length (mm) a_0 (4)	Ligament size (mm) $d - a_0$ (5)	Span (mm) b (6)	Ultimate strength σ_u (MPa) (7)	Fracture energy $\mathcal{G}_F$ (N/m) (8)
A	5	500 × 150 × 110	37.5	112.5	315	1.24	120.7
B	5	500 × 200 × 110	50	150	425	0.96	149.1
C	3	750 × 300 × 110	75	225	630	0.87	194.2
D	2	1,250 × 400 × 110	100	300	875	0.80	256.9
E	2	2,250 × 600 × 110	150	450	1,315	0.735	434.1

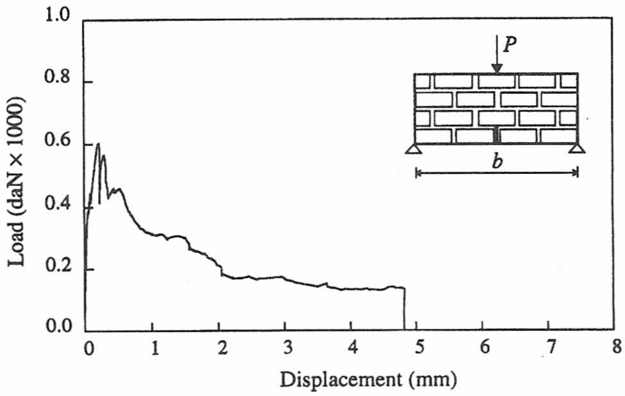


FIG. 8. Typical Experimental Load-Displacement Curve

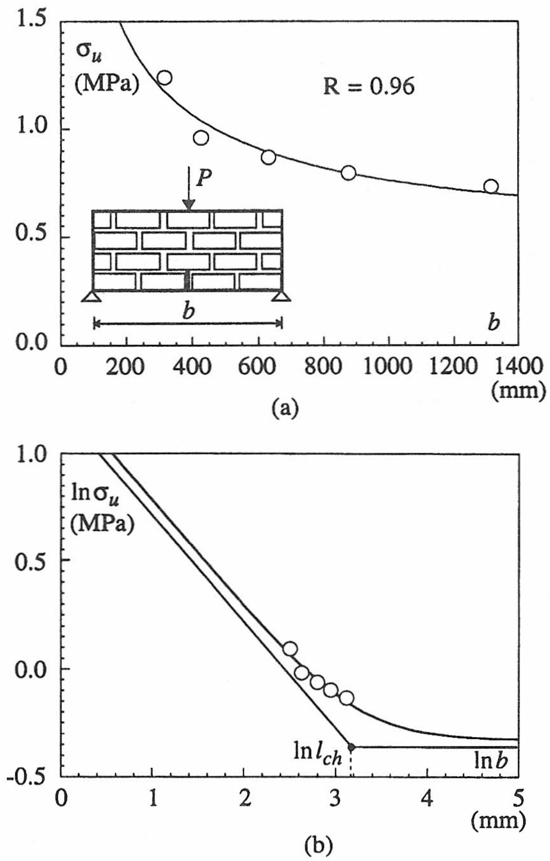


FIG. 9. Application of MFSL to Experimental Flexural Strength Data

under the complete load-displacement curve (Fig. 8) divided by the ligament area, according to the RILEM (1985) recommendation, and properly taking into account the influence of the specimen weight.

ANALYSIS OF TEST RESULTS BY MEANS OF MULTIFRACTAL SCALING LAWS

A strong size dependence of the mechanical quantities has been detected in the three-point bending tests on the brick masonry walls, as can be deduced from Table 1. The scaling behavior of the tensile strength σ_u and of the fracture energy $\mathcal{G}_F$ has been consequently analyzed by means of the MFSL (1*a*) and (1*b*). The nonlinear Levenberg-Marquardt algorithm has been used in order to perform best-fitting of the experimental data (Marquardt 1963).

In Fig. 9(*a*), the MFSL is applied to the strength data; from the best-fitting procedure, the asymptotic value f_t of the masonry strength, valid for infinite size, is equal to 0.477 MPa while the characteristic internal length l_{ch} is equal to 1,672 mm. The regression coefficient, quantifying the goodness of fit, results in $R = 0.96$. Note that the value of the asymptotic strength is less than 1/3 of the smallest specimen's value; therefore, the decrease of strength occurring in real-sized structures has to be adequately taken into account when extrapolating the values obtained from laboratory-sized specimens. In the bilogarithmic diagram, represented in Fig. 9(b), the upward concavity of the data clearly emerges, which confirms the transition towards an ordered regime at larger scales (Carpinteri 1994b).

Note that, according to the multifractal hypothesis, a constant value of strength can be determined only for sizes larger than l_{ch}, where the heterogeneities in the masonry composition can be considered homogenized. In the spirit of the mean-field approach, an RVE could be selected in the masonry with size equal to or larger than l_{ch}, and only structures "sufficiently" larger than the RVE could be considered statistically homogeneous.

The application of the MFSL to the fracture energy data is shown in Fig. 10(a). Nonlinear regression yields the following values of the parameters: $\mathcal{G}_F^\infty = 3{,}438$ N/m and $l_{ch} = 161{,}720$ mm. In this case, a lower regression coefficient ($R = 0.61$) is obtained if compared to the previous case. A stronger influence of the constitutive heterogeneity is provided by the toughness data, which seem to increase with size according to a very steep slope. As it can be deduced from Fig. 10(b), where a wider extent of the size range is considered, the experimental values of the masonry fracture energy are set in the highly heterogeneous (fractal) regime, apparently controlled by a fractality even more disordered than the Brownian regime.

Consequently, the asymptotic value $\mathcal{G}_F^\infty$ of the fracture en-

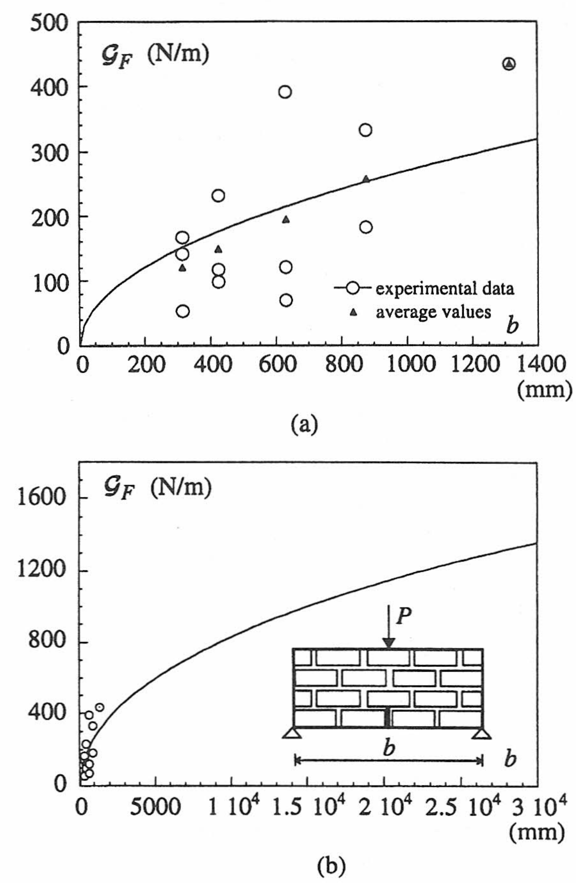


FIG. 10. Application of MFSL to Experimental Fracture Energy Data

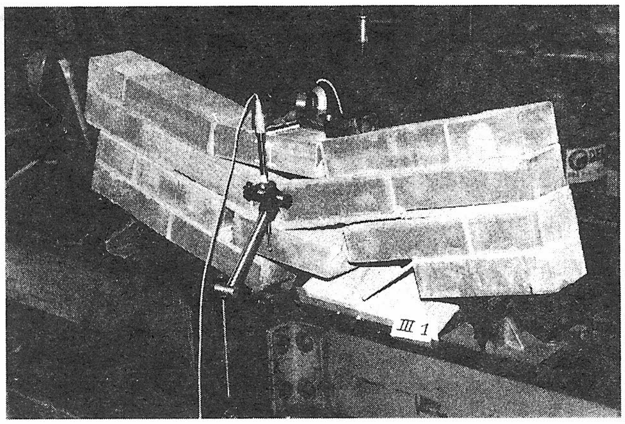


FIG. 11. Bending Collapse of Masonry Panel

ergy is comparatively enormous with respect to the experimental data. It can be argued that, in the tested masonry walls, the fracture process zone (FPZ) is not sufficiently developed due to the relatively small structural size, and therefore only a small amount of energy comes into play during the fracture process. A minimum structural size, related to the value of l_{ch}, is required to fully activate the FPZ, and only beyond that size, a constant asymptotic value of the fracture energy would be determined.

On the other hand, the best-fitting parameters appear overestimated in the case of toughness, the value of l_{ch} being two orders of magnitude larger than the corresponding value determined from the scaling of strength. Such a discrepancy is certainly due to the narrow range of sizes that have been tested. The nominal strength σ_u, determined according to the linear elastic bending theory, represents a well-defined quantity, since the mechanism of the bending failure is already fully activated even in the smallest specimens, as it has been detected during the tests (Fig. 11). On the contrary, the dissipation mechanism is not completely developed in these small masonry walls, and the transition to a scale-independent regime seems to occur only at much larger scales.

CONCLUSIONS

The applicability of the "fractal" fracture mechanics to brick masonry walls has been discussed. Three-point bending tests have been carried out on five different sizes, yielding the peculiar scaling of the mechanical quantities involved in the fracture process. The following conclusions may be drawn:

1. The mechanical behavior of brick masonry structures is highly controlled by the constitutive heterogeneity (disorder) of the material.
2. Statistical homogeneity in masonry can be obtained only at very large scales. Only when the structural size is much larger than an appropriate internal characteristic length can the mean-field approach be adopted in order to characterize this composite material.
3. The multiscale cooperation in the breaking process has to be taken into account. This aspect can be modelized by means of multifractal distributions of the mechanical properties, related to the multifractal topologies involved in the phenomenon.
4. Fracture mechanics parameters like the fracture energy $\mathcal{G}_F$ appear as fundamental quantities in the characterization of masonry structures behavior. The behavior under high strain gradients (seismic loads), where the classical stress criteria are clearly inadequate, cannot be properly understood without information on the toughness characteristics.
5. The size scaling of the main mechanical properties can be interpreted by means of two multifractal scaling laws, which allow for the extrapolation of the strength and toughness values from the experimental data to the real-sized structures, as well as for the determination of the transition scale l_{ch} beyond which the effect of the heterogeneity vanishes.
6. Nevertheless, a wider size range has to be tested, in order to describe, in a more precise manner, the entire scaling behavior. In the case of fracture energy, larger masonry panels should particularly be tested, in order to provide the full activation of the energy dissipation mechanism.

ACKNOWLEDGMENTS

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APPENDIX I. REFERENCES

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APPENDIX II. NOTATION

The following symbols are used in this paper:

a = size of material characteristic defect;
a_0 = notch length in three-point bending (TPB) specimens;
b = structural reference size (span of TPB specimens);
d = depth of TPB specimens;
f_t = asymptotic strength valid for infinite sizes;
$\mathcal{G}_F$ = fracture energy according to RILEM (1985) definition;
$\mathcal{G}_F^\infty$ = asymptotic fracture energy valid for infinite sizes;
l_{ch} = characteristic internal length of masonry;
M_u = ultimate bending moment at failure;
R = correlation coefficient of best fitting procedure; and
t = thickness of TPB specimens.