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Power Scaling Laws and Dimensional Transitions in Solid Mechanics

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Abstract—Physicists have often observed a scaling behaviour of the main physical quantities during experiments on systems exhibiting a phase transition. The main assumption of a scaling theory is that these characteristic quantities are self-similar functions of the independent variables of the phenomenon and, therefore, such a scaling can be interpreted by means of *power-laws*. Since a characteristic feature of phase transitions is a catastrophic change of the macroscopic parameters of the system undergoing a continuous variation in the system state variables, the phenomenon of fracture of disordered materials can be set into the wide framework of critical phenomena. In this paper new mechanical properties are defined, with non integer physical dimensions depending on the scaling exponents of the phenomenon (i.e. the fractal dimension of the damaged microstructure, or the exponent of a power-constitutive relation), which turn out to be scale-invariant material constants. This represents the so-called renormalization procedure, already proposed in the statistical physics of random process. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION: FRACTURE AS A CRITICAL PHENOMENON

The concept of *phase transition* represents a fundamental aspect in the physics of condensed matter [1]. From everyday experience, let us examine, for instance, water and its various phases: solid (ice), liquid, and gas (steam). A change in temperature or pressure allows one to observe the familiar transitions from one phase to another. An interesting aspect to be highlighted is the sudden character of the transitions that may occur after a small shift in temperature or in some other parameter: this leads to the definition of *critical phenomena* and *critical points*. In correspondence to the critical points, which are fixed points in the phases space, two or more different phases are in unstable equilibrium, in the sense that, at the interfaces, regions belonging to different phases coexist. The system then evolves unexpectedly into one of the phases, by just a very small variation of the controlling parameters.

The remarkable observation that a characteristic feature of phase transitions is a discontinuous (catastrophic) change of the macroscopic parameters of the system undergoing a continuous variation in the system state variables, leads us to set the phenomenon of fracture of disordered materials into the wider framework of critical phenomena [2]. We shall focus on the two major quantities governing the fracture behaviour of disordered materials: the ultimate tensile strength σ_u and the fracture energy \mathcal{G}_F .

The occurrence of criticality in the phenomenon of rupture of solids is evident: both fracture energy and ultimate tensile strength may then be regarded as critical parameters. The former represents the specific energy (per unit area) required for the separation of a continuum into two parts and is equal to the critical value of the strain energy release rate: this separation is nothing other than a transition from a sound phase (monolithic) to a new one, characterized by a matter discontinuity. This kind of transition reveals the coexistence of broken bonds with the sound ones and shows a remarkable physical instability (brittle

fracture). The latter parameter, σ_u , is usually defined as the maximum tensile stress that a material can undergo. As shown in Fig. 1, where a classical stress–strain curve for a disordered material (like concrete) is represented, the ultimate tensile strength can be considered a critical parameter, since the transition from the ascending branch to the softening one is characterized by a mechanical instability.

From a microstructural point of view, this transition means a bifurcation of the microcracks behaviour: at strains less than the critical one, virtually no microcrack propagates to a larger scale, thus the body is macroscopically sound, whereas, above the critical strain, microcracks suddenly coalesce, leading to the macroscopic fracture of the body (catastrophic failure). The phenomenon of fracture is, therefore, not completely random nor completely correlated. The local breaking rule obviously depends on the (randomly distributed) material properties, but also on the surrounding stress–strain state which becomes increasingly important as the external load increases (i.e. when the critical load is approached). The post-critical behaviour, in particular, corresponds to a highly correlated development of the fracture surface.

It is interesting to note the powerful analogy of the ultimate tensile strength with the percolation threshold in a percolation model, which is a powerful tool employed to simulate many critical phenomena [3]. We could in fact define the *correlation length* ξ as the length of the interaction range among the microcracks, at a given applied stress: as the applied stress is increased, a value of stress is reached at which the correlation length becomes infinite and a macroscopic failure occurs. Moreover, at this point, the correlation length is not only infinite, but it remains infinite under scaling transformations. Since, in the theory of critical phenomena, the occurrence of the correlation length being equal to infinity plus the aforementioned invariance property mean that the system is at a critical point, we could state that the ultimate strength is the *critical stress*, i.e. the value at which the behaviour changes catastrophically from stable to unstable (in the sense of Drucker's instability).

The main difference between the phenomenon of rupture of solids and the classical phase transitions is the irreversibility of the transition. For example, in the case of water transition from liquid to gas, a certain amount of thermal energy is given to the molecules, in order to raise their own energy. On the other hand, the latter can be recovered, just by obtaining the inverse transition (gas to liquid). On the contrary, when energy is given (by loading) to a solid body up to a complete fracture into two parts, there is no way to

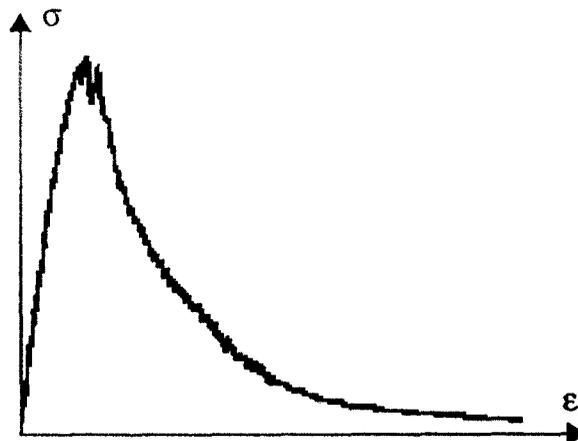


Fig. 1. Constitutive relation for a strain-softening material.

recover the potential (elastic) energy stored in the material since it is all dissipated for breaking the bonds and developing the fracture surface. Therefore, we have the trivial conclusion that it is impossible to restore continuity in a broken solid.

2. SCALING THEORY AND POWER-LAWS

Physicists have often observed a scaling behaviour of the main physical quantities during experiments on systems exhibiting a phase transition [1]. The main assumption of a scaling theory is that these characteristic quantities (for example, free energy \mathcal{F} or magnetic field \mathcal{B}) are self-similar functions of the independent variables of the phenomenon (for example, temperature related to the critical point $T - T_c$ or magnetization M):

$$\mathcal{B} = C|T - T_c|^\beta. \quad (1)$$

These scaling laws are nowadays believed to be exact in ‘asymptotia’, that is, in the asymptotic limit to the critical point ($T \rightarrow T_c$) [4].

From a mathematical point of view, scaling implies a power-law dependence of the physical quantities on the independent variables. Power-laws have the generic form:

$$y = Cx^\beta \quad (2)$$

and therefore are described by two parameters, the *amplitude* C and the *exponent* β . The amplitude is generally not of intrinsic interest, since it depends on the choice made for the definitions of the physical quantities. The exponent, on the other hand, depends on the process itself, i.e. on the self-similar rule that governs the scaling.

A crucial point of critical phenomena is that the same scaling exponents describe the asymptotic behaviour of several different physical processes, whereas the amplitudes sensitively depend on the particular system being examined. Therefore, different physical phenomena belong to the same class of *universality*, that is, they are characterized by the same scaling properties.

Stanley first pointed out the deep analogy underlying non-integer (fractal) dimensions and scale-invariant phase transitions, even recalling renormalization group (RG) theory: “Since one can introduce fractal dimensions that play the role of critical exponents, and since relations among the dimensions play the role of scaling laws, it is natural to ask: What about Renormalization Group? [1].

It should be noted here that any power-law distribution is mathematically equivalent to a fractal distribution. Just as the perimeter P of a fractal coastline scales with the yardstick length l according to a non-integer exponent α , ($P \sim l^{(1-\alpha)}$), the exponents characterizing the power-law scaling of the critical quantities involved in a phase transition are usually found to have *non-integer* values. This implies a deep analogy: the common underlying rule is self-similarity, which is behind this non-standard scaling in both cases. The scale-invariance of a physical system at a critical point has been clearly recognized either by experimental or theoretical approaches. At the critical point the system shows large fluctuations (regions belonging to one of the two phases) with no characteristic size. The above scale-invariance forms the basis of the renormalization group theory, which has been successfully applied to the analysis of continuous phase transitions, providing the calculation of the critical exponents, which confirmed to be non-integer numbers, and the determination of the so-called *fixed points* in the phase diagrams. If fractals represent the geometry of disorder, renormalization group represents the physical backbone behind it: the two theories are thus so intimately related that we could say that fractals are the geometrical aspect of RG theory.

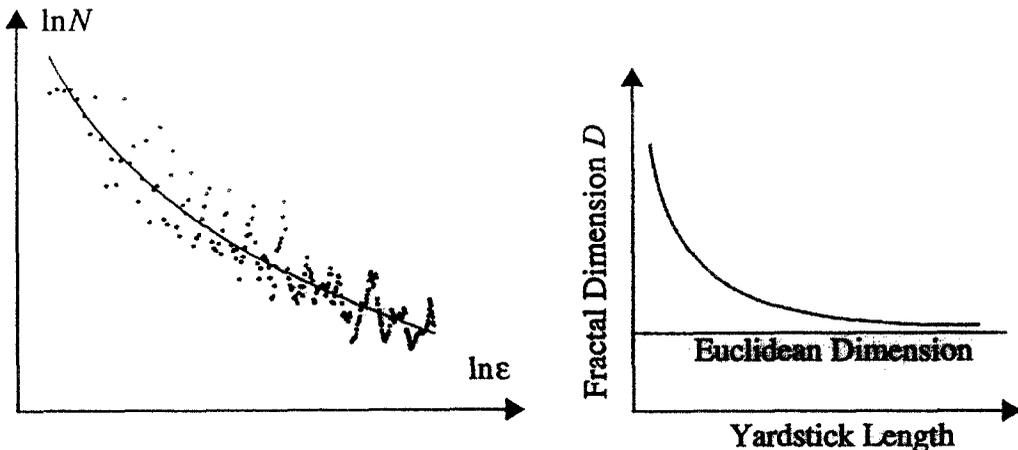
3. TOPOLOGICAL SCALING: FRACTAL FRACTURE SURFACES

3.1. Fractality of material microstructure

Fractal geometry deals with highly disordered morphologies, like rough interfaces or microcracked structures. Recent physical theories [5] show that natural fractal surfaces arise as a joint consequence of randomness (due to the geometrical and constitutive disorder of the microstructure) and optimality (that is, optimal energy expenditure). Experimental evidence of random self-similarity (*fractality*) over a broad range of scales has often been reported in recent literature: fractured surfaces of steel [6], molybdenum [7], natural rocks [8] and concrete [9] were shown to share fractal properties, at least between an upper scale bound, related to the macroscopic dimension of the object, and a lower scale bound which could be set, depending on the material microstructure, equal to the smallest grain size or to the atomic size of particle.

Random self-similarity implies *topological scaling* in the sense that (statistically) similar morphology appears in a wide range of magnifications of the fracture surface. Besides randomness, another fundamental aspect of fractals in nature is the inevitable crossover to homogeneous behaviour at the larger scales. This transition from a fractal scaling (characterized by a non-integer topological dimension) to a 'smooth' regime (where the Euclidean dimension holds) can be referred to as *self-affine* scaling [10] if only the two limiting aspects are taken into account, or as *geometrical multifractality* if the whole continuous transition is considered [11]. In the case of concrete fracture surfaces, such a transition is represented by the diagrams in Fig. 2(a, b), where the compass method has been applied to a fracture profile. A curve rather than a straight line fits the data satisfactorily, since a unique slope (= unique fractal dimension) cannot be associated with the profile, but a transition to Euclidean dimension is present.

On the other hand, the fractal nature of fractured surfaces produces a dimensional increment with respect to the number 2. The mechanical interpretation of the fractal dimension being non-integer and greater than 2 could be that the dissipation of energy during the development of a fracture surface is intermediate between surface energy



(a)

(b)

Fig. 2. Geometrical multifractality for a fracture profile.

dissipation [which is the linear elastic fracture mechanics (LEFM) hypothesis [12]] and volume energy dissipation (which is the classical approach of strength of materials theories, limit analysis and damage mechanics).

Defining new mechanical properties with physical dimensions depending on the fractal dimension of the damaged material microstructure, represents the so-called *renormalization* procedure, already utilized in the statistical physics of random processes [13]. RG approaches have also been successfully developed in the context of fluid turbulence, as a systematic way of progressively eliminating the effects of the smallest eddies and replacing their equivalent effect by an effective turbulent viscosity [4]. In this context, it is also worth mentioning the well-known Kolmogorov approach to turbulence, where fractional power-laws of the double-velocity correlation functions come into play. It is nowadays believed that these scaling laws can be derived from the fractal (*Cantorian*) characteristics of the underlying space–time framework [14].

3.2. Renormalized fracture energy

The main physical purpose of the RG method is always that of extracting macroscopic phenomenology from microscopic models. More precisely, the goal is to determine the large-scale (or long-term) behaviour of a given system, and to establish the existence of universality close to the transition. In this way it is possible to obtain the so-called *universal* properties, i.e. scale-invariant material constants.

A renormalization group transformation can be applied to the scaling of fracture energy \mathcal{G}_F due to the topological scaling of the dissipation space. Let us model this ‘surface’ by means of a deterministic *invasive* fractal (Fig. 3). For the sake of simplicity, let us refer to

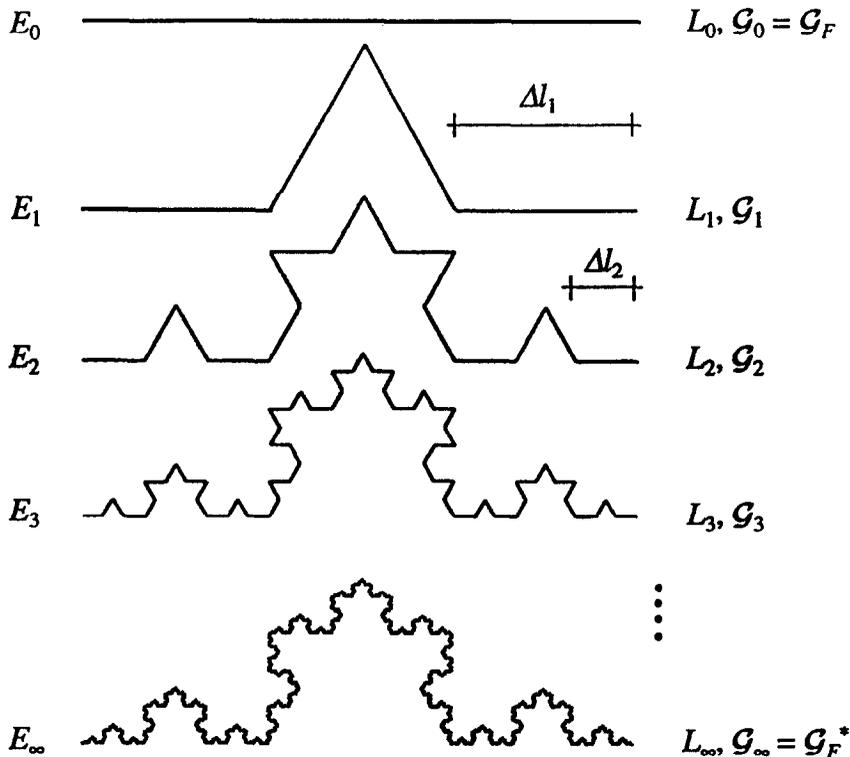


Fig. 3. Renormalization of fracture energy over an invasive fractal dissipation space.

a bidimensional structure (a profile), that is, the intersection of the fractal surface with a vertical plane. The fractal dimension, which is larger than 2 in the case of the surface, satisfies $1 < D_g < 2$ in the case of the profile, and the self-similarity, losing the random character, becomes a deterministic property.

Let us consider the fractal profile at various magnification scales (Fig. 3). We note that, refining the scale of observation, the energy dissipation involves, at each step, a larger amount of nominal 'surface'. At level E_0 (the macrolevel) this space looks like a smooth surface (length = L_0). At level E_1 it can be readily noted that the nominal length $L_1 = 4\Delta l_1 = (4/3)L_0 > L_0$, while at level E_2 it becomes $L_2 = 16\Delta l_2 = (16/9)L_0 > (4/3)L_0$ and so on. Ideally the length tends to infinite in the limit represented by the proper fractal set (E_∞). Such a scaling involves the property of fractal sets to be non-rectifiable sets, that is, *non-measurable by means of the canonical dimensions*. It is, therefore, a remarkable conceptual misunderstanding, from our point of view [15], to deduce from fractality a trivial numerical increment of the nominal fracture area, as has been suggested by many authors. On the contrary, this self-similar increment ($L_{k+1} > L_k$), which comes from the disordered material microstructure as well as from the formation of multiple microcracks inclined with respect to the direction of the macrocrack, provides non-integer Hausdorff dimensions of the space on which the energy dissipation occurs.

At each scale of observation, a fictitious fracture energy \mathcal{G}_n can be associated, which is considered as the energy dissipated at the scale n during the formation of the unit crack area. On the other hand, the total dissipated energy ΔW is a 'macro-parameter', in the sense that it is independent of the observation scale, since it has to satisfy the global energy balance with the critical strain energy release rate. Thus, the following equalities must hold:

$$\Delta W = \mathcal{G}_0 L_0 = \mathcal{G}_1 N_1 \Delta l_1 = \mathcal{G}_2 N_2 \Delta l_2 = \dots = \mathcal{G}_n N_n \Delta l_n = \dots, \tag{3}$$

where N_n is the number of elements at the level n , each of length Δl_n ($N_n \Delta l_n = L_n$). Generalizing to any couple of subsequent scales:

$$\mathcal{G}_n N_n \Delta l_n = \mathcal{G}_{n+1} N_{n+1} \Delta l_{n+1}, \tag{4}$$

and then, rearranging the above equation:

$$\mathcal{G}_n = \frac{N_{n+1}}{N_n} \frac{\Delta l_{n+1}}{\Delta l_n} \mathcal{G}_{n+1}. \tag{5}$$

Equation (5) represents the RG transformation, relating nominal fracture energy at scale n to the same quantity at scale $n + 1$. Let p be the ratio of similarity ($p = 3$ for the von Koch curve in Fig. 3):

$$\frac{\Delta l_{n+1}}{\Delta l_n} = \frac{1}{p}. \tag{6}$$

From the fundamental relations of fractal geometry, the fractal dimension of the profile can be deduced as:

$$D_g = \frac{\ln\left(\frac{N_{n+1}}{N_n}\right)}{\ln p}, \tag{7}$$

where $D_g = 1.26$ in the case of the von Koch curve.

In its simplest form, a RG-like equation for some essential physical quantity ϕ can be interpreted as stating that the variation of ϕ under an infinitesimal scale transformation ($\ln \Delta l_n$) depends on ϕ itself [16]:

$$\frac{\partial \phi}{\partial(\ln \Delta l_n)} = f(\phi), \tag{8}$$

and therefore this yields that the nominal fracture energy at each scale strictly depends on the scale length itself:

$$\mathcal{G}_n = \mathcal{G}(\Delta l_n). \tag{9}$$

From eqns (6) and (9), eqn (5) becomes:

$$\mathcal{G}(\Delta l_n) = \frac{N_{n+1}}{N_n} \frac{1}{p} \mathcal{G}\left(\frac{\Delta l_n}{p}\right). \tag{10}$$

Moreover, in the context of critical phenomena (scale invariance), we can express the asymptotic dependence on the scale as a power-law relation:

$$\mathcal{G}(\Delta l_n) \approx (\Delta l_n)^x. \tag{11}$$

Equation (10) can therefore be rearranged to give:

$$(\Delta l_n)^x = \frac{N_{n+1}}{N_n} p^{-(x+1)} (\Delta l_n)^x, \tag{12}$$

which holds for any couple of subsequent scales due to the scale invariance at the critical point (*fixed point* in the RG transformation). This kind of universality corresponds, in our case, to a monofractal space. Equation (12) can be solved in x , in order to extract the critical scaling exponent:

$$x = \log_p \left(\frac{N_{n+1}}{N_n} \right) - 1 = \frac{\ln \left(\frac{N_{n+1}}{N_n} \right)}{\ln p} - 1, \tag{13}$$

and finally, from eqn (7):

$$x = D_q - 1. \tag{14}$$

Since the fractal dissipation space possesses a dimension strictly larger than the topological one ($D_q = 1.2619$ from the fractal in Fig. 3 is obtained), if we denote by d_q the fractional dimensional increment with respect to the Euclidean space ($d_q = D_q - 1$), we obtain:

$$\mathcal{G}(\Delta l_n) \approx (\Delta l_n)^{d_q}. \tag{15}$$

This relation expresses the scaling behaviour of the nominal fracture energy with respect to lengths. In the general tridimensional case, all the length dimensions have to be increased by one and the critical exponent becomes equal to $D_q - 2$. Equation (15) is still valid, since the dimensional increment d_q is always a positive number between 0 and 1.

In order to obtain a scale-independent physical quantity, we are therefore forced to abandon the usual physical dimensions of $\mathcal{G}_F([F][L]^{-1})$ and to move to the ‘renormalized fracture energy’ characterized by non-integer physical dimensions:

$$\mathcal{G}_F^* = [F][L]^{-(1+d_q)}. \tag{16}$$

It is interesting to note [17] that the same result can be obtained starting from the hypothesis of scale invariance and considering a sequence of scales of observation where the first scale of observation is the *macroscopic* one [A_1 being the cross-sectional area and $\mathcal{G}_1 = \mathcal{G}_F$ the conventional fracture energy], while the asymptotic scale of observation is set

as the *microscopic* (fractal) one [$A_\infty = A^*$ being the *measure*, in the Hausdorff sense, of the fractal set and \mathcal{G}_F^* the *renormalized fracture energy* defined in eqn (16)]:

$$\Delta W = \mathcal{G}_F A_1 = \mathcal{G}_2 A_2 = \dots = \mathcal{G}_F^* A^* \tag{17}$$

From the definition of Hausdorff measure [18], if b is a characteristic dimension of the cross-section, the following relations hold:

$$\begin{aligned} A_1 &\approx b^1, \\ A^* &\approx b^{2+d_g}, \end{aligned} \tag{18}$$

and therefore, equating the extreme members of the scaling cascade in eqn (17), the scaling relation of eqn (15) is recovered in a slightly different form:

$$\mathcal{G}_F \sim \mathcal{G}_F^* b^{d_g} \tag{19}$$

or, in a logarithmic form:

$$\ln \mathcal{G}_F = \ln \mathcal{G}_F^* + d_g \ln b \tag{20}$$

which implies a linear scaling in the bilogarithmic diagram (see Fig. 4). This last equation represents the well-known size effect on fracture energy, as has been detected experimentally by many authors [11]. The Hausdorff non-integer *measurability* is therefore equivalent to the *renormalizability* of the physical quantities, thus confirming that the RG and the fractal approach essentially involve the same physics [1].

3.3. Griffith fractal crack: dimensional transition for the stress-intensity factor

The non-integer physical dimensions of the *renormalized fracture energy*, previously obtained by means of renormalization group transformations, can be also determined by means of a simple energy-balance approach, following the way traced by Griffith [12] and Irwin [19].

We start from the assumption of a fractal crack of projected length $2a$, in place of the classical Griffith smooth crack (Fig. 5). For the sake of simplicity, let us assume a deterministic geometrical fractal profile, instead of a random one: this does not make our

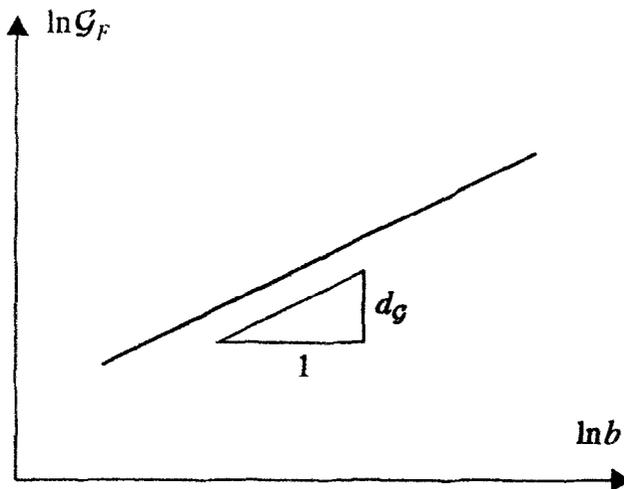


Fig 4. Size effect on nominal fracture energy.

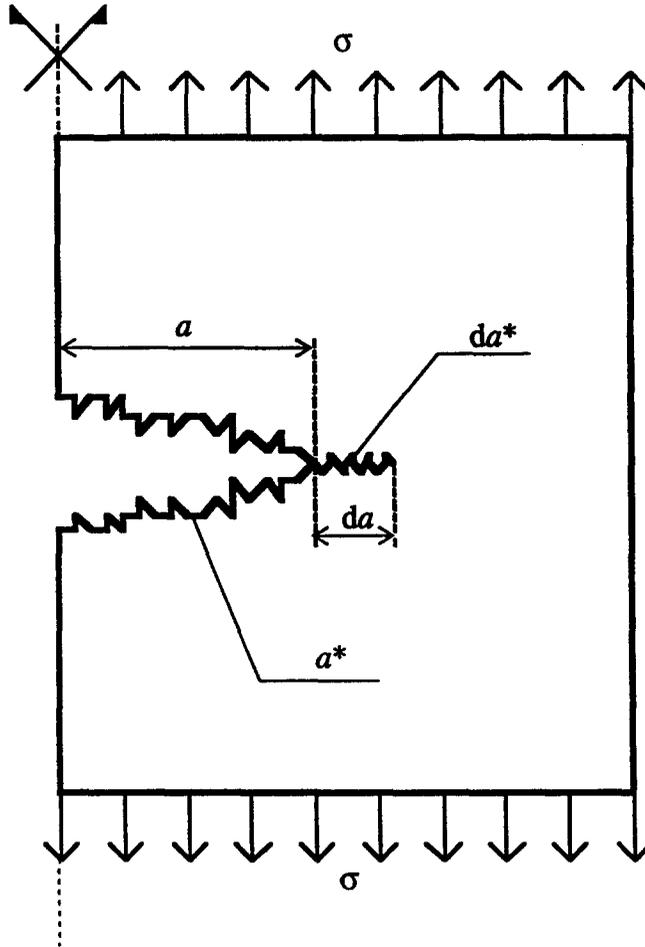


Fig 5. Stress–intensity factor at the tip of a fractal crack.

treatise lose generality. Recalling the basic relations from fractal geometry, if we consider a fracture profile as a fractal set a^* , with projected length a and fractal dimension $D_g = d_g + 1$, the following fundamental relation holds:

$$N_n r_n^{D_g} = a^{D_g}, \tag{21}$$

where $n = 1, 2, \dots, \infty$ is the iteration scale (Fig. 3), a is the projected length, also called the *initiator*, r_n is the scaling factor of lengths at scale n and N_n is the number of segments at scale n . Equation (21) states the self-similarity of the set throughout the whole scale range. The nominal length L_n of the fractal profile, measured at scale n , that is, computed by means of a yardstick length r_n walked along the profile, is equal to $N_n r_n$ and, from eqn (21), to:

$$L_n = a^{(1+d_g)} r_n^{-d_g}. \tag{22}$$

One can easily verify that, in the limit of $r_n \rightarrow \infty$, that is, in the limit of the smallest scales of observation, this fictitious length tends to infinity. The above circumstance is a fundamental feature of fractal geometry: all the attempts made in order to define any physical quantity over a finite length value of a fractal set must be considered as a conceptual mistake. More and more irregularities are computed as the observation scale

decreases, and no geometric tangent can be univocally associated to any point of the profile: this expresses the well-known characteristic of fractal sets to be *non-differentiable* (in a classical sense) mathematical objects.

The fundamental Griffith relation of energy balance, at the critical point of unstable crack propagation, states:

$$\frac{dW_E}{da} = \frac{dW_S}{da} \quad (23)$$

where the first term represents the elastic energy release rate due to the propagation, equal to $2da$ in the horizontal projection, of the pre-existing fractal crack (Fig. 5), and dW_S is the total energy dissipated on the developing fractal crack, due to the breaking of the material bonds and the coalescence of microcracks. The elastic energy release is a macroscopic and global parameter, which means that it is independent of the observation scale: it is therefore not sensitive to disorder, so that fractality does not come into play in its definition. On the contrary, as already shown in the last section, dW_S represents the energy directly dissipated on the fractal set, and thus the *nature* of this dissipation is intimately controlled by the disordered microstructure. On the other hand, due to the requested energy balance, eqn (23), the global quantity dW_S holds the usual physical dimensions of energy ($[F][L]$). In the case of a smooth crack (Griffith's original solution):

$$\frac{dW_E}{da} = \frac{d}{da} \left(\frac{\sigma^2 \pi a^3}{E} \right) = \frac{2\pi \sigma^2 a}{E}; \quad \frac{dW_S}{da} = 4\gamma. \quad (24)$$

where $\gamma([F][L]^{-1})$ represents the surface energy dissipated on each face of the opening crack. By substituting eqns (24) in the balance relation (23), and defining the 'fracture energy' \mathcal{G}_F of the material as $\mathcal{G}_F = 2\gamma$, the well-known Griffith criterion for brittle fracture is obtained:

$$\sigma \sqrt{\pi a} \cong \sqrt{(\mathcal{G}_F E)}. \quad (25)$$

which can be expressed, following Irwin's demonstration [19], in terms of the stress-intensity factor, as $K_I \cong K_{IC}$, where K_{IC} is the *critical stress-intensity factor* or the *fracture toughness* of the material. Note that, in the case of smooth cracks, the singular stress field at the crack tip is characterized by the power of the singularity equal to $1/2$, it being controlled by a stress-intensity factor with dimensions $[F][L]^{-3/2}$.

In the case of a fractal crack, we assume that, at each scale of observation n , a nominal *microscopic* fracture energy \mathcal{G}_n is dissipated along a nominal length L_n : the rate of energy dissipated *at this scale* as each crack tip moves of the infinitesimal projected quantity da (Fig. 5), can therefore be given by:

$$\left(\frac{dW_S}{da} \right)_n = 2 \frac{d(\mathcal{G}_n L_n)}{da}. \quad (26)$$

In order to obtain a scale-independent value, since the fractal set is, by definition, a limit concept, (i.e. the set of points corresponding to the scale of observation tending to zero), and recalling eqn (22), we can write:

$$\frac{dW_S}{da} = 2 \times \lim_{n \rightarrow \infty} \left(\frac{d(\mathcal{G}_n L_n)}{da} \right) = 2(1 + d_\mathcal{G}) a^{d_\mathcal{G}} \left[\lim_{n \rightarrow \infty} \left(\mathcal{G}_n r_n^{-(1+d_\mathcal{G})} \right) \right]. \quad (27)$$

Note that the expression inside the square brackets is indeterminate since it results in a form $0 \times \infty$. The key of the above procedure is to solve the limit in eqn (27) by invoking the renormalization group theory:

$$\lim_{n \rightarrow \infty} (\mathcal{G}_n r_n^{-(d_g)}) = \mathcal{G}_F^*, \quad (28)$$

which can be recognized as the re-interpretation of eqn (15). From the above relation we obtain again the non-integer dimensions $([F][L]^{-(1+d_g)})$ of fracture energy depending on that of the dissipation fractal space. Inserting eqn (28) into eqn (27), the following expression can be obtained:

$$\frac{dW_s}{da} = 2(1 + d_g)\mathcal{G}_F^* a^{d_g}. \quad (29)$$

The procedure of eqns (26)–(29) represents a physically justified methodology in order to bypass the formal non-differentiability of fractal sets. The rate of change of a physical quantity defined on a fractal space is provided by an ordinary analytical derivative and it is possible to extract a scale-independent (renormalized) value due to the powerful RG transformations, synthesized by eqn (28).

It is interesting to note that eqn (29) can be provided, as already shown [17], by directly considering a fractal fracture energy \mathcal{G}_F^* with non-integer dimensions, and taking the simple derivative of a^* with respect to a , where $a^* = a^{(1+d_g)}$ is the ‘measure’ (in the Hausdorff sense) of the fractal set. This means that the *mass* (in a topological sense) of the fractal set can be univocally (that is, scale-independently) obtained only by measuring it by means of length raised to $(1 + d_g)$. In this context one can simply write:

$$\frac{dW_s}{da} = 2\mathcal{G}_F^* \frac{d(a^*)}{da} = 2(1 + d_g)\mathcal{G}_F^* a^{d_g}, \quad (30)$$

which exactly agrees with eqn (29). This procedure could be called a ‘projective derivative’ [17], and arises from the relation between fractal and classical coordinates. Classical coordinates are obtained when the fractal space is smoothed by means of balls of radii greater than some critical value, that is, looking at the fractal ‘from outside’. As can be readily seen in eqn (30), the intersection of a regular fractal a^* of dimension $D_g = 1 + d_g$ with a Δ -manifold ($\Delta = 1$ in our case) is another fractal, the *projective derivative* of a^* , with dimension $D_g - 1$.

Based on eqn (29) or (30), the energy balance relation (23) in the case of fractal cracks yields:

$$\sigma^2 \pi a^{(1-d_g)} = (1 + d_g)\mathcal{G}_F^* E, \quad (31)$$

which can be written in terms of the (*fractal*) *stress–intensity factors* as:

$$(K_I^*)^2 = (K_{IC}^*)^2. \quad (32)$$

The *fractal stress–intensity factor*, therefore, presents the following physical dimensions:

$$[K_I^*] = [F][L]^{-(3+d_g)/2}, \quad (33)$$

which represents the dimensional transition of this physical parameter, corresponding to the dimensional transition of fracture energy in eqn (16). Generalizing the well known Irwin’s solutions, the near-tip elastic stress field can be written as [19]:

$$\sigma_{ij} = K_I^* r^{-(1-d_g)/2} f_{ij}(\vartheta), \quad (34)$$

where the power of the singularity strictly depends on d_g . The dimensional transition provides an attenuation of the fracture localization, involving an attenuation of the stress singularity and, macroscopically, a more ductile behaviour for disordered materials. When $d_g = 0$, we find again the classical relations of Griffith and Irwin. When $d_g = 1$, as a limit case, we find that the stress-singularity at the crack tip vanishes and K_I^* assumes the

physical dimensions of stress [17]. Therefore, no localization occurs, as if the energy were dissipated in a volume ($D_g = 3$).

An immediate comparison between eqns (24) and (30) shows another important consequence of the fractality: while in the case of a smooth crack the fracture energy is independent of a , being constant during crack propagation, in the presence of fractal cracks it increases with a , following a power-law with fractional exponent equal to d_g (Fig. 6). That is, if the nominal fracture energy is related to the renormalized one, by comparing eqns (24) and (30), one obtains: $dW_c/da = 2G_F \approx 2G_F^* a^{d_g}$. This provides the following considerations.

- (1) The presence of disorder introduces non-linearity in the fracture behaviour of a linear elastic material. What is considered as a material constant during crack propagation, turns out to be an increasing function of the crack length, thus implying that *the crack resistance grows during the propagation*. The analogy with the *R-curve* theory is extensive, even if in that theory the non-linearity comes from the constitutive laws of the material.
- (2) Equation (30) is the 'shell' of the monofractal size-effect behaviour as has been interpreted by Carpinteri [15]: in fact, in the double-logarithmic diagram $\log G_F$ vs $\log b$, where b is a reference dimension of the structure, the nominal fracture energy increases with a slope equal to d_g (Fig. 4).

3.4. Renormalized tensile stress

Many researchers have investigated the microcracking properties of disordered materials such as concrete, in order to highlight the damage evolution due to progressively increasing tensile stresses. By means of three-dimensional acoustic emission techniques, the fracture process zone has been shown to exist before the peak load.

From this point of view, the rarefied resisting section in correspondence to the critical cross-section can be modeled by a stochastic *lacunar* fractal set of dimension D_o , with $1 < D_o \leq 2$. The same RG procedure of Section 3.2 can be implemented over this self-similar set, observing that now the fractal dimension is smaller than the topological one.

Let us assume, for simplicity, a deterministic fractal set (*middle-third Cantor set*), and let us consider the two-dimensional case. Thus, the fractal dimension drops to a value $0 < D_o \leq 1$. As is shown in Fig. 7, where the fractal cross-section is represented at various

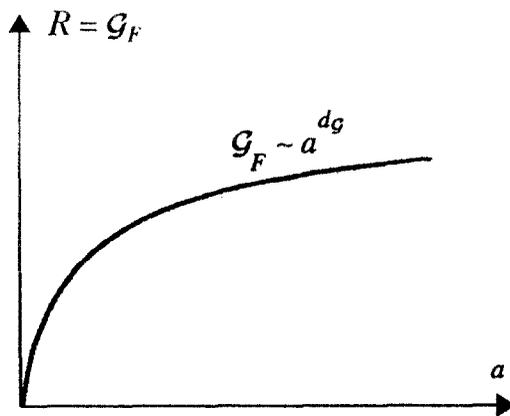


Fig 6. R-curve behaviour from a fractal approach.

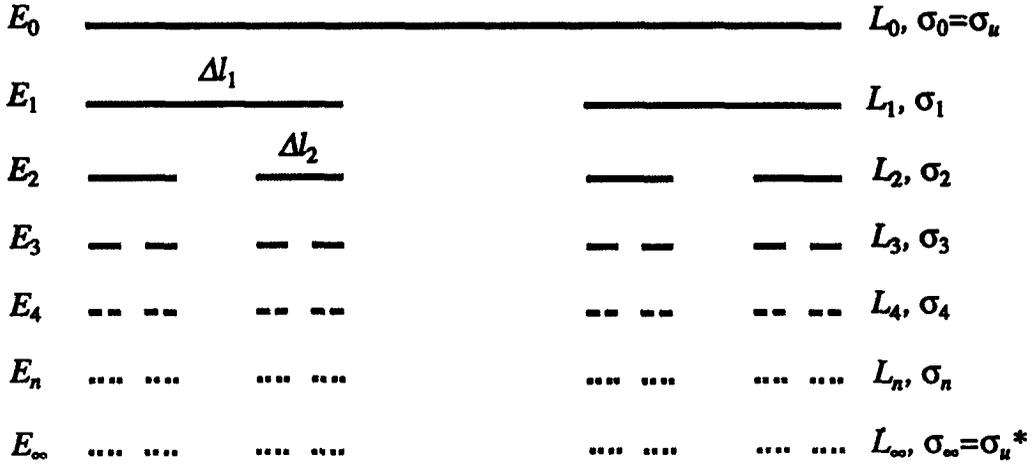


Fig 7. Renormalization of tensile strength over a lacunar fractal resisting section.

magnification scales, we note that, refining the scale of observation, the resisting section, at each step, is represented by a smaller amount of nominal ‘surface’. At level E_0 (the macrolevel) this space looks like a smooth surface (length = L_0) and the whole resisting section appears to transmit the stresses through the body. At level E_1 it is apparent that the resisting section is reduced due to damage to $L_1 = 2\Delta l_1 = (2/3)L_0 < L_0$, while at level E_2 it becomes $L_2 = 4\Delta l_2 = (4/9)L_0 < (2/3)L_0$ and so on, E_n ideally tending to zero in the limit represented by the proper fractal set (E_∞). This self-similar cross-sectional weakening is due to the presence of pores, voids, cracks and inclusions at all length scales and its extent depends also on the evolving nature of the material damage. Such a scaling involves again, as in the case of the (*invasive*) von Koch curve (Fig. 3), the property of these sets of *being not measurable at the canonical dimensions*.

At each scale of observation, a fictitious micro-stress σ_n can be associated, which is considered as the stress carried at the scale n . On the other hand, the total external force F is a ‘macro-parameter’, in the sense that it is independent of the observation scale, since it has to satisfy the global force equilibrium condition. Thus, the following equalities must hold:

$$F = \sigma_0 L_0 = \sigma_1 N_1 \Delta l_1 = \sigma_2 N_2 \Delta l_2 = \dots = \sigma_n N_n \Delta l_n = \dots, \tag{35}$$

where N_n is the number of elements at level n , each of length Δl_n ($N_n \Delta l_n = L_n$). Generalizing to any couple of subsequent scales:

$$\sigma_n = \frac{N_{n+1}}{N_n} \frac{\Delta l_{n+1}}{\Delta l_n} \sigma_{n+1}. \tag{36}$$

Equation (36) represents the RG transformation, relating micro-stress at scale n to the same quantity at scale $n + 1$. Let p be the ratio of similarity ($p = 3$ for the Cantor set in Fig. 7 also): from the fundamental relations of fractal geometry, the fractal dimension of the resisting section projection can be deduced as:

$$D_\sigma = \frac{\ln \left(\frac{N_{n+1}}{N_n} \right)}{\ln p}, \tag{37}$$

where $D_\sigma = 0.63$ in the case of the triadic Cantor set (Fig. 7).

Following the same RG arguments of Section 3.2, that is, expressing the asymptotic dependence of σ_n on the scale as a power-law relation ($\sigma_n \approx (\Delta l_n)^d$), the hypothesis of scale invariance finally yields:

$$\nu = D_c - 1 = -d_o, \quad (38)$$

which represents the critical exponent for the scaling behaviour of the ultimate tensile stress with respect to the length:

$$\sigma_n(\Delta l_n) \approx (\Delta l_n)^{-d_o}. \quad (39)$$

In the general tridimensional case all the length dimensions have to be increased by one and the critical exponent becomes equal to $D_c - 2$. Eqn (39) is obviously still valid, since the absolute value of the topological dimensional decrement d_o is always a number between 0 and 1.

In order to obtain a scale-independent physical quantity, we are therefore forced to abandon the usual physical dimensions of σ_u ($[F][L]^{-2}$) and to move to the 'renormalized tensile strength' characterized by non-integer physical dimensions:

$$[\sigma_u^*] = [F][L]^{-(2-d_o)}. \quad (40)$$

Again, the same result can be obtained starting from the hypothesis of scale invariance and considering a sequence of scales of observation where the first scale of observation is the *macroscopic* one (A_1 being the cross-section area and $\sigma_1 = \sigma_u$ the conventional tensile strength), while the asymptotic scale of observation is set as the *microscopic* (fractal) one [$A_\infty = A^*$ being the *measure*, in the Hausdorff sense, of the fractal set and σ_u^* the 'renormalized tensile strength' defined in eqn (40)]:

$$F = \sigma_u A_1 = \sigma_2 A_2 = \dots = \sigma_u^* A^*. \quad (41)$$

From the definition of Hausdorff measure [18], if b is a characteristic dimension of the cross-section, the following relations hold:

$$\begin{aligned} A_1 &\approx b^2, \\ A^* &\approx b^{2-d_o}, \end{aligned} \quad (42)$$

and, therefore, equating the extreme members of the scaling cascade in eqn (41), the scaling relation of eqn (39) is recovered in a slightly different form:

$$\sigma_u \approx \sigma_u^* b^{-d_o}. \quad (43)$$

or, in a logarithmic form:

$$\ln \sigma_u = \ln \sigma_u^* - d_o \ln b, \quad (44)$$

which implies a linear scaling in the bilogarithmic diagram, with slope d_o (Fig. 8). The last equation represents the well-known decreasing size effect on tensile strength, as has been detected experimentally by many authors [20]. Again, it is shown that the Hausdorff concept of non-integer *measurability* is equivalent to the concept of *renormalizability* of the physical quantities.

Carpinteri [11] correlated the critical exponents of tensile strength and fracture energy, by means of dimensional analysis arguments, postulating that the reversal of the physical roles of toughness and strength is absurd. The dimensional limit for both the exponents seems therefore to be equal to 1/2, corresponding, in the case of the fracture surfaces, to an *invasive Brownian surface* (with Hausdorff dimension 2.5, [18]) and, in the case of the resisting sections, to a rarefied (*Cantorian* or *dust-like*) set with Hausdorff dimension 1.5. Note the remarkable analogy between the Cantorian model of *lacunar* material ligaments

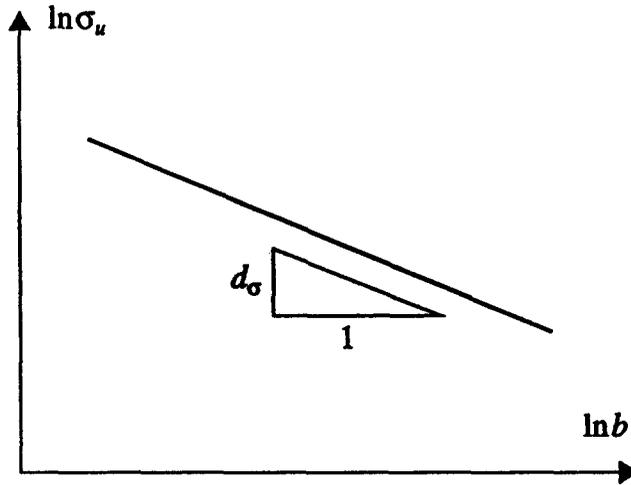


Fig 8. Size effect on nominal tensile strength.

(dimension = 1.5) and the Kolmogorov theory of turbulence, where a 3/2 power-law comes into play in the scaling [14]. It can be argued that a connection must exist between all the chaotic dissipative phenomena.

Thereby, a theoretical Brownian disorder yields: $d_\sigma + d_q < 1$ [11]. More generally, very intricate media could exceptionally present $d_q > 1/2$ (overhangs) and, therefore, $d_\sigma < 1/2$ (stronger resisting section), the previous inequality being valid in any case.

4. CONSTITUTIVE SCALING: POWER-LAW HARDENING MATERIAL

It is interesting, at this point, to note that the dimensional transition, previously deduced for the toughness parameters, can be obtained by using a power-law hardening constitutive relation for the material, in the way traced by Carpinteri [21]. The power-law hardening Ramberg–Osgood stress–strain law (Fig. 9) can be expressed by:

$$\tilde{\varepsilon} = \tilde{\sigma}^n, \tag{45}$$

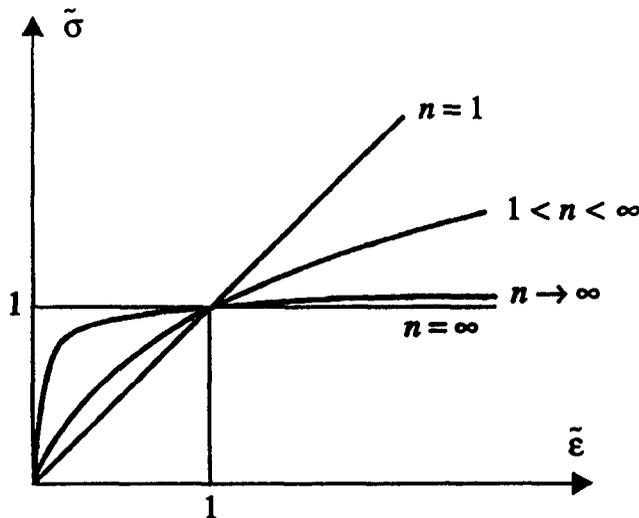


Fig 9. Ramberg–Osgood power-law hardening constitutive relation.

where the exponent n can be any real number between 1 and $+\infty$. Solving the eigenvalue problem connected with the differential equation governing the Airy stress function near the crack tip, it is possible to define a *plastic stress intensity factor*, K_I^P , which is directly connected to the J -integral and presents physical dimensions depending upon the hardening exponent n :

$$[K_I^P] = [F][L]^{-(2n-1)/(n+1)} \quad (46)$$

The plastic stress field close to the tip of the crack is then described by:

$$\sigma_{ij} = K_I^P r^{-\beta} f_{ij}(\theta) \quad (47)$$

where $\beta = 1/(n+1)$. The stress-singularity then proves to be a function of the hardening exponent. In correspondence of the two limit cases of linear elastic material ($n = 1$) and rigid-perfectly plastic material ($n \rightarrow \infty$), two extreme situations can be recognized.

- (1) When $n = 1$ (linear elastic material), K_I^P has the canonical dimensions $[F][L]^{-3/2}$, and the stress-singularity is nothing other than the LFM singularity [$\beta = 1/2$ in eqn (47)].
- (2) When $n \rightarrow \infty$ (rigid-plastic material), K_I^P tends to assume the physical dimensions $[F][L]^{-2}$, i.e. the dimensions of a stress. As regards the plastic fracture toughness, K_{IC}^P , it can be said that this fracture parameter coincides with the yield strength σ_y , and it seems that the considered model cannot predict a crisis different from plastic collapse [21]. Moreover, the stress-singularity tends to disappear together with the localization of the energy dissipation ($\beta \rightarrow 0$).

Increasing n from 1 to infinity, i.e. increasing the non-linearity of the material, a transition becomes evident from a brittle fracture collapse to a plastic flow collapse. The failure behaviour of a material then shows the same trend towards a larger ductility either by increasing the hardening exponent n in a power-law hardening constitutive relation, or by increasing the fractal dimension of the fracture surface. We wish to stress again that this more ductile behaviour of a disordered material does not involve a numerical increment of the conventional fracture resistance parameters (\mathcal{G}_F or K_{IC}), but is due to the new anomalous dimensions assumed by these quantities when the fractality of the disordered microstructure is considered.

The strong analogy between the results of our renormalization group procedure and the assumption of a *power-law constitutive relation* for the material can be easily explained from a mathematical point of view. Applying to a linear elastic material the RG transformations on the energy dissipation space means to define the mechanical quantities on a *power-law geometrical field*, which is called a 'fractal'. On the other hand, the Ramberg–Osgood constitutive relation means a *power-law mechanical behaviour* on a classical Euclidean geometrical field is assumed for the material. The geometrical power-law seems then to be equivalent to a mechanical power-law. It is the *self-similarity*, which is behind both approaches, that gives rise to the non-integer dimensions of fracture energy \mathcal{G}_F^* and fracture toughness K_{IC}^* .

5. GEOMETRICAL SCALING: STRESS-SINGULARITY AT THE VERTEX OF RE-ENTRANT CORNERS

5.1. Dimensional transition for the generalized stress–intensity factor

A similar dimensional transition, with consequent stress-singularity attenuation and reduction of the rate of decrease in strength with size, was pointed out by Carpinteri [22] for linear elastic materials with re-entrant corners (Fig. 10).

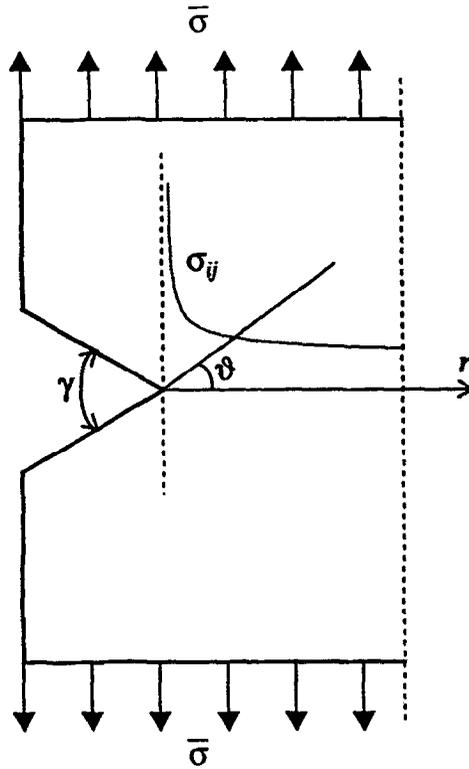


Fig 10. Stress-singularity at the vertex of a re-entrant corner.

The problem of the determination of the stress-intensification at the vertex of a re-entrant corner was tackled and solved by Williams [23] in 1952; experimental and numerical confirmations of these results were provided by Leicester [24] and Walsh [25]. Carpinteri [22] observed that it is the *power* of the stress-singularity and not the geometry of crack, structure and load, which determines the rate of strength decrease by increasing the size-scale: therefore, based upon Williams’ solution, the size-effect transition slope was related, by means of dimensional analysis concepts, to the variation of the stress-intensity factor physical dimensions, obtained by varying the value of the angle γ .

Williams [23] obtained his solution by means of a *power-law series expansion* of the Airy stress function, written in polar coordinates as indicated in the following equation:

$$\Phi(r, \vartheta) = \sum_n r^{\lambda_n+1} f_n(\vartheta), \tag{48}$$

where λ_n is the eigenvalue corresponding to the angular function $f_n(\vartheta)$ (eigenvector). Based upon this assumption, Williams proved that, when both notch surfaces are stress-free, the symmetrical stress field at the notch tip is given by:

$$\sigma_{ij} = \bar{K}_I r^{-\alpha} S_{ij}^{(\gamma)}(\vartheta), \tag{49}$$

where the power α of the stress-singularity is provided by the eigenequation:

$$(1 - \alpha) \sin(2\pi - \gamma) = \sin[(1 - \alpha)(2\pi - \gamma)] \tag{50}$$

and ranges between 1/2 (when $\gamma = 0$) and zero (when $\gamma = \pi$), as illustrated by the diagram in Fig. 11.

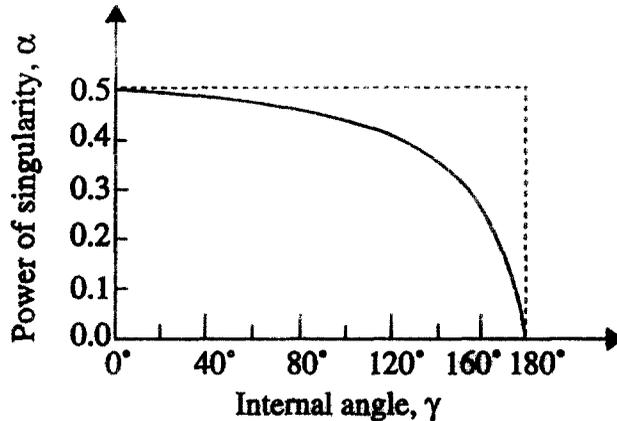


Fig 11. Power of stress-singularity as a function of the amplitude of a re-entrant angle.

If Buckingham's theorem for physical similitude and scale modelling is applied and stress and linear size are assumed as fundamental quantities, it is possible to obtain a *generalized stress-intensity factor*:

$$\bar{K}_1 = \bar{\sigma} b^\alpha \bar{f}(a/b), \quad (51)$$

with physical dimensions ranging from $[F][L]^{-3/2}$ when $\alpha = 1/2$ ($\gamma = 0$) to $[F][L]^{-2}$ when $\alpha = 0$ ($\gamma = \pi$). In eqn (41), $\bar{\sigma}$ is the far-field constant stress and $\bar{f}(a/b)$ represents a non-dimensional geometrical factor depending on the ratio between the crack size a and the structural characteristic size b , whose determination is not of interest in this context. The stress of failure σ_f is achieved when the \bar{K}_1 factor is equal to its critical value \bar{K}_{1c} :

$$\sigma_f = \bar{K}_{1c} b^{-\alpha} \frac{1}{\bar{f}(a/b)}. \quad (52)$$

In the bilogarithmic diagram the strength ($\ln \sigma_f$) results in a linear decreasing function of the scale parameter ($\ln b$), with negative slope α . Therefore, when $\gamma \rightarrow \pi$, i.e. when $\alpha \rightarrow 0$, any scale effect vanishes and the straight line becomes horizontal while, when $\gamma \rightarrow 0$, i.e. when $\alpha \rightarrow 1/2$, the classical $-1/2$ slope, corresponding to sharp cracks, is obtained (Fig. 12).

The remarkable dimensional transition of the *generalized stress-intensity factor* \bar{K}_1 , provided by the variation of the re-entrant corner angle γ , corresponds exactly to the dimensional transition of the *fractal stress-intensity factor*, obtained by varying the fractal dimension of the fracture surface [see eqn (33)]. While in the latter case the non-usual physical dimensions of fracture toughness come from the assumption of energy dissipation over non-Euclidean (fractal) spaces, thus generalizing Griffith's criterion for brittle fracture, in the former the dimensional transition for \bar{K}_1 comes from the stress-singularity at the vertex of a Euclidean re-entrant corner. In both cases a power-law distribution governs the dimensional transition: a power-law (self-similar) fractal field over which energy dissipation is supposed to occur [eqn (30)], or a power-law singular stress-field [eqn (49)] around the notch tip.

5.2. Dimensional transitions as a product of fractional derivatives

The geometrical scaling behaviour observed for the stress-intensity factor in presence of re-entrant corners can be also analyzed by means of fractional derivatives. It is widely

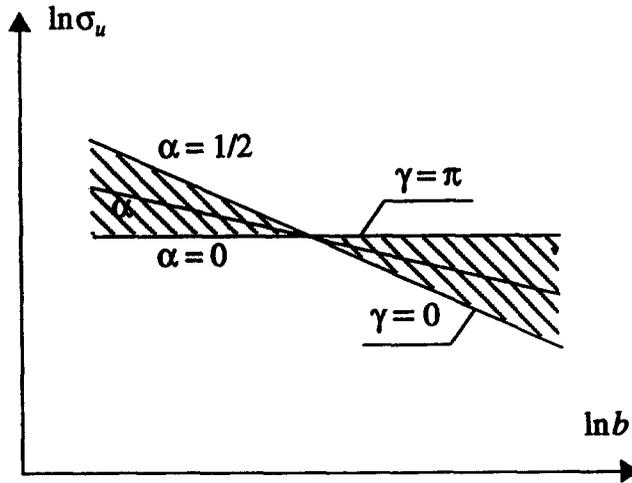


Fig 12. Attenuation of the size-scale transition in the case of re-entrant corners.

believed, nowadays, that the power-law scaling of a physical phenomenon implies fractional differentiation. On the other hand, non-integer scaling exponents could be analytically deduced in a consistent manner through fractional differential equations involving continuous functions.

The so-called ‘fractional calculus’, that is, the study of mathematical operators able to make derivatives and integrals of *any order* (not necessarily positive integers), is increasingly spreading outside pure mathematics towards several physical applications, such as rheological and viscoelastic material behaviour [26], energy exchanges at fractal interfaces and non-linear damping of dynamical systems [27]. In all the aforementioned applications, the mathematical fundamental aspect of the fractional *differintegral* operator clearly emerges as the *analytical continuity*, that is, the capacity of continuously interpolating between the extreme conditions of a physical phenomenon, covering, in a concise and elegant manner, its whole constitutive spectrum.

The fundamental definition of the fractional operator is due to Riemann and Liouville [see, for example, Oldham and Spanier [28]] who, generalizing the classical Cauchy integral formula, proposed the following definition:

Let α be a positive real number, f a continuous function of x , and let a be any fixed number. By definition, the ‘ α -order derivative’ of f is given by:

$${}_a\mathcal{D}_x^\alpha(f(x)) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \left(\int_a^x f(t)(x - t)^{n-\alpha-1} dt \right), \quad (53)$$

where Γ is the Euler–Gamma function (the analytical extension of the factorial integer operator $n!$). Note that the α -order derivative of f can be obtained as the classical n -order derivative of the $(n - \alpha)$ -order integral of f , where $n > \alpha$. This definition is independent of n , provided that $n > \alpha$. Thus, n is usually set as the smallest integer larger than α .

As can be immediately deduced from the above definition, the α -order fractional derivative of a continuous function $f(x)$ depends on the lower integration limit a . This is not the case, however, when α is a positive integer, as in this case [i.e. $n = \alpha$ in eqn (53)] the classical Cauchy formula is recovered. This is the most important distinction between conventional and fractional differentiation. While, in fact, the classical derivative is a ‘local’

operator, univocally determined in a point x , the fractional derivative holds a 'non-local' or *integral* character (it is always, actually, an integral operator).

Particularly interesting, for our purposes, is the fractional differentiation of a constant function. Let $f(x) = C$. If $1 > \alpha > 0$, it results:

$${}_a^R D_x^\alpha f(x) = \frac{C(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \tag{54}$$

and if the lower limit of integration is set as $a = 0$:

$${}_0^R D_x^\alpha f(x) = \frac{d^\alpha C}{dx^\alpha} = \frac{Cx^{-\alpha}}{\Gamma(1-\alpha)}. \tag{55}$$

Therefore, unlike the conventional derivatives of a constant function being identically zero, the α -order fractional derivative of a constant function is different from zero. On the contrary, such a derivative is a continuous function of x , singular at $x = a$, and *the rate of singularity α coincides with the order of differentiation*. For any positive integer $\alpha = n \geq 1$, the Gamma function in eq. (54) would become infinite, and thus ${}_a^R D_x^n(C)$ would reduce to the conventional derivative, which is obviously identically zero. Any scaling exponent can be derived, therefore, from a proper differentiation of a constant function.

The dimensional transitions of the stress-intensity factor can, therefore, be homogeneously described by means of the fractional operator which naturally allows one to obtain the singular stress field at the vertex of the re-entrant corner, by simple differentiation (of non-integer order) of the far-field $\bar{\sigma}$ (constant stress).

Based on dimensional analysis, the stress field at the tip of the re-entrant corner can be expressed as:

$$\sigma_{ij} \sim b^\alpha \frac{d^\alpha(\bar{\sigma})}{dr^\alpha} S_{ij}^{(\gamma)}(\vartheta), \tag{56}$$

where the derivative has to be taken with the lower limit of integration $a = 0$, since the crack tip is the source of the disturbance and, therefore, can be considered as the origin of the r -axis.

By means of eqn (56), all the possible situations can be described by varying the order of fractional derivation. In particular, if $\alpha = 0$, then $\sigma_{ij} = \bar{\sigma}$, corresponding to the vanishing of the singularity ($\gamma = \pi$). On the other hand, if $\alpha = 1/2$, then the LEFM singularity is obtained:

$$\sigma_{ij} \sim (\bar{\sigma} b^{1/2}) r^{-1/2} S_{ij}^{(\gamma)}(\vartheta) \sim K_I r^{-1/2} S_{ij}^{(\gamma)}(\vartheta). \tag{57}$$

The 1/2-order derivative thus yields the LEFM solution for a Griffith crack. Increasing the order of differentiation (that is, decreasing the internal angle of the corner), corresponds to increasing the 'disturb effect' of the defect on the surrounding constant stress field (Fig. 13).

Similar arguments have been exploited by Engheta in order to determine the (singular) electrostatic potential field in the proximity of a sharp conducting wedge [29]. In the latter approach, on the other hand, no dimensional transition for the physical quantities due to the fractional operator has been revealed, which, from our point of view, is the fundamental aspect to be highlighted.

6. CONCLUSIONS

In the previous sections, non-integer physical dimensions have been determined for the main mechanical quantities governing the tensile failure of disordered materials. Either if

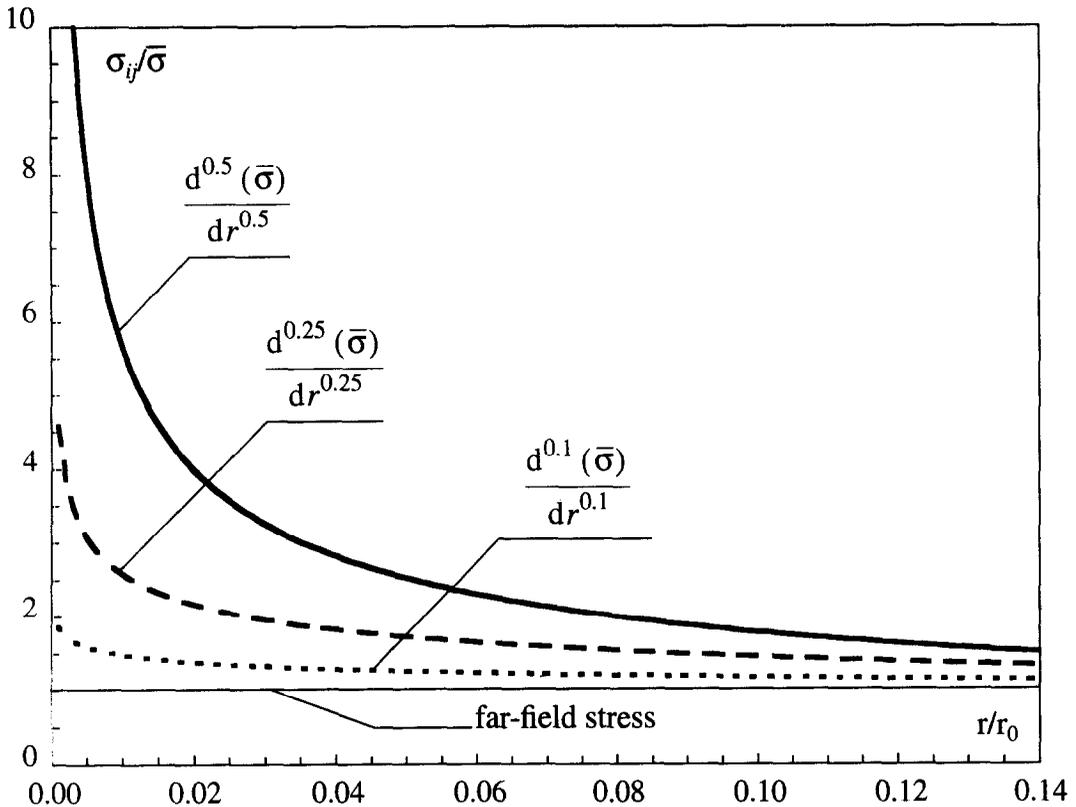


Fig 13. Singular stress field as a result of fractional derivation of the constant far-field.

this is obtained by means of the renormalization procedure (in the case of *renormalized* fracture energy \mathcal{G}_F^* and of *renormalized* tensile strength σ_u^*), or by means of fractal projective derivatives (in the case of the fractal stress-intensity factor K_I^*), or, finally, by means of dimensional analysis arguments and fractional differentiation (in the case of the plastic stress-intensity factor K_I^P and of the generalized stress-intensity factor \bar{K}_I), these anomalous dimensions come from the scaling (power-law) behaviour of the corresponding quantities.

The assumption of non-integer physical dimensions allows one to determine physical quantities that are invariant with respect to the length scale (universality). Moreover, a continuous transition can be detected for these scaling exponents, from the conventional dimensions to a totally different dimensionality, which implies a completely different nature of the fracture phenomenon, from the microscopic as well as from the macroscopic point of view. In particular, the dimensional transition of the toughness parameters (K_I or \mathcal{G}_F), accompanied by the disappearance of the stress-singularity, represents the remarkable transition from a brittle breaking behaviour to a ductile failure, and explains the vanishing of size-scale effects, as ductility increases.

It could be concluded that many non-classical problems in mechanics can be adequately tackled if the canonical physical dimensions are abandoned. In particular, just as turbulence may be the intermediate link between classical and non-classical fluid dynamics, chaotic fractals may provide the connection between classical and quantum mechanics [30].

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