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TECHNICAL NOTES

DISCUSSIONS

their products can be neglected. Based on this assumption, the finite element equation may be linearized to, but not including, the order of u^2.

Two essential operations may affect the solution accuracy of an incremental procedure. One is the establishment of the incremental (linearized) equations of equilibrium for the finite element, and therefore for the structure, from which the incremental displacements can be solved. The other is the calculation and updating of element forces based on the solved displacements. The paper has noted that proper account of the rigid body effects is central to the success of these operations in analysis of beam structures. It should be added that, in an incremental nonlinear analysis, the effect of large displacements or rotations is included through updating of the nodal coordinates in a step-by-step fashion. This is clearly one advantage of the incremental procedure. The user need not deal with the highly nonlinear problems, but only with the linearized (nonlinear) ones.

Errata. The following changes should be made to the original paper:

Page 1408, Eq. 10*a*: Should read $e_{xx} = u'_{xC} - yu''_{yC}$

Page 1408, Eq. 10*d*: Should read $\eta_{xy} = \frac{1}{2}(-u'_{xC}u'_{yC} + yu'_{yC}u''_{yC})$

Page 1410, Eq. 18*c*: Should read $M_z = -M_{zA}(1 - i) + M_{zB}i$

Page 1412, Eq. 24: Should read

$$[k_g]\{u\}_r = \left\langle -\frac{M_{zA} + M_{zB}}{L}\theta_r, -F_{xB}\theta_r, 0, \frac{M_{zA} + M_{zB}}{L}\theta_r, F_{xB}\theta_r, 0\right\rangle^T$$

Analysis of Mixed-Mode Fracture in Concrete[a]

Discussion by A. Carpinteri,[3] Member, ASCE, and S. Valente[4]

A very interesting aspect of the paper, in the writers' opinion, lies in the snap-back behavior of the mixed-mode crack propagation. The snap-back load-displacement branch may be captured experimentally only if the loading process is controlled by a monotonically increasing function of time, e.g., the crack mouth opening or sliding displacement. On the other hand, the snap-back load-displacement branch may be captured numerically only if the loading process is controlled by a monotonically increasing function of the crack length. Examples of such functions are provided by the arc-length control scheme (Riks 1979; Ramm 1981; Crisfield 1981), and by the indirect displacement control scheme (de Borst 1986). Both these techniques use a displacement norm as controlling parameter. On the other hand, as a mono-

[a]November, 1987, Vol. 113, No. 11, by Jan G. Rots and René de Borst (Paper 21979).
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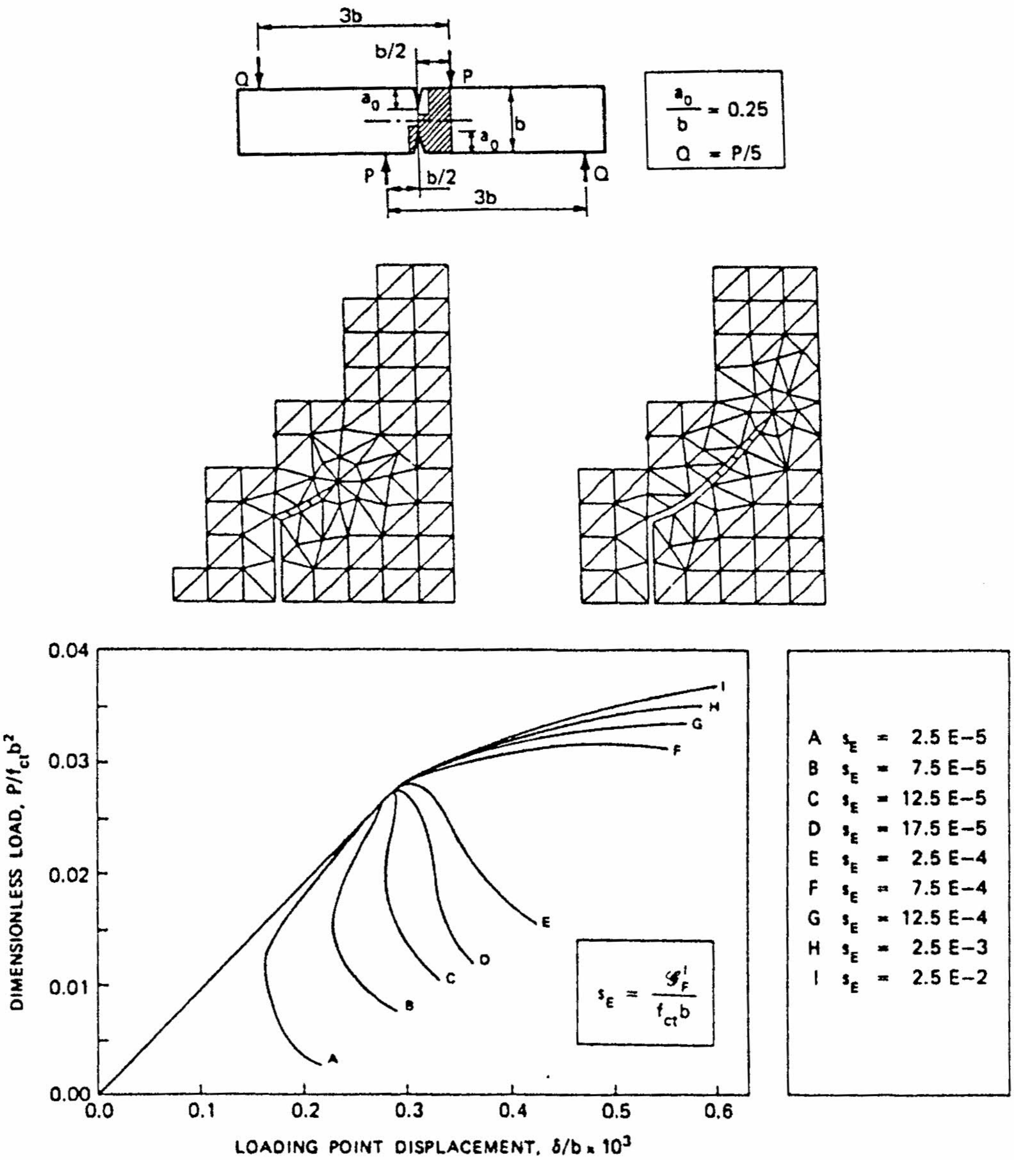

FIG. 16. Mixed-Mode Size Effects

tonically increasing function of the crack length, the first writer used the crack length itself (Carpinteri 1984; Carpinteri et al. 1985). The same numerical procedure, based on discrete cohesive crack modeling, was then extended from mode I to mixed-mode crack propagation by the writers.

Whereas for mode I only node untieing was applied to simulate crack propagation, a continuous modification of the mesh was required for the mixed-mode interelement crack propagation. The numerical response of a four-point shear specimen is presented in Fig. 16. A finite element rosette was translated and rotated at each noncollinear crack growth step. Fig. 16 shows two different steps of the crack propagation process. The presence of the cohesive forces is shown by the connections. At the first step the cohesive zone is missing and the load P producing the ultimate tensile stress at the crack tip is computed. Such a value, P, together with the related loading point displacement, δ, gives the first point of the P-δ curve. The fictitious crack tip propagates by a predefined length Δa in a direction orthogonal to the maximum circumferential stress. Then, the load P producing the ultimate tensile stress at the updated fictitious crack tip and the related deflection δ are computed at each step in order to obtain the subsequent points of the curve.

Some dimensionless load-displacement diagrams obtained numerically for the four-point shear specimen are displayed in Fig. 16, by varying the number s_E. For $s_E \lesssim 15.0 \times 10^{-5}$, the curves reveal a snap-back instability.

The mixed-mode brittle fractures are interpretable in terms of snap-back instability. Their virtual development may be analyzed by the previously presented fictitious crack length control scheme, whereas LEFM describes only the instability condition. In the discussers' opinion, the fictitious crack length control scheme allows to overcome the uncertainties of the displacement norm control schemes in problems where failure modes are highly localized, because the crack length is always and unequivocally a monotonically increasing quantity during the irreversible fracture process.

Appendix. References

Carpinteri, A. (1984). "Interpretation of the Griffith instability as a bifurcation of the global equilibrium." *NATO Advanced Research Workshop on Application of Fracture Mechanics to Cementitious Composites*, S. P. Shah, ed., Martinus Nijhoff, 287–316.

Carpinteri, A., Di Tommaso, A., and Fanelli, M. (1985). "Influence of material parameters and geometry on cohesive crack propagation." *RILEM Int. Conf. on Fracture Mech. of Concr.*, F. H. Wittmann, ed., Elsevier, Lausanne, Switzerland, 117–135.

Closure by Jan G. Rots[5] and René de Borst[6]

The writers would like to thank Carpinteri and Valente for their interest in the paper. They broaden the scope of our paper by presenting a discrete crack analysis of a double-notched specimen. Recently, Rots (1988) has carried out a similar analysis. As in our present paper and in previous contributions by the second author (de Borst 1986, 1987), Carpinteri and Valente obtain snap-back behavior of a specimen composed of strain-softening material for a certain range of (realistic) values of the softening modulus.

Inasmuch as the value of the brittleness factor S_E (or equivalently, the mode I fracture energy G_f^I) influences the steepness of the drop of the load after the limit point, mode II effects also have a profound influence on the postfailure behavior. The writers have shown that improper relations for mode II behavior may lead to an overstiff structural response in the postfailure regime. Unfortunately, Carpinteri and Valente only consider mode I effects and thus completely bypass the effects of the shear stress-strain response in a crack beyond the peak load.

As noted in the discussion, snap-back behavior requires a careful guidance of the solution. When turning points with a horizontal and vertical tangent are encountered along the equilibrium path, solution procedures that involve the path length as an additional variable must be employed in order that the entire equilibrium path can be tracked. The writers have emphasized (de Borst 1987) that in case of strain localization keeping track of the path length

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in a reduced displacement space, e.g., a space that involves only the displacements to the left and right of the notch (either crack-opening displacement or crack-mouth-sliding displacement) is superior to controlling the path length in the space of all degrees-of-freedom. Bearing this in mind, the crack length will probably also be a good choice for a controlling parameter. Still, the writers have the opinion that there are some problems inherent in using the crack length as controlling parameter:

- The crack length does not provide a clear and transparent correlation with experiments. Since the crack-opening displacement and the crack-mouth-sliding displacement are often used as feedback signals in servocontrolled testing, the use of these parameters seems more natural for controlling the numerical simulations.
- It is unclear how the crack length can be used in smeared finite element computations since the crack length is then undefined.
- The definition of crack length is ambiguous in case of more cracks in a specimen. In fact, the problem analyzed by Carpinteri and Valente poses exactly this problem. Their beam has two notches and upon first loading crack propagation will start from both notches. Near the peak load one of the crack will continue to propagate while the other crack will be arrested (Rots et al. 1987a, b). Consequently, the success of the use of the crack length as controlling parameter depends crucially on the proper selection of the active crack. This implies that the suggested advantage of the crack length as controlling parameter over for instance the crack-opening displacement disappears.

Appendix. References

de Borst, R. (1987). "Computation of post-bifurcation and post-failure behavior of strain-softening solids." *Comp. & Struct.*, 25, 211–224.
Rots, J. G., Hordijk, D., and de Borst, R. (1987a). "Numerical simulation of concrete fracture in 'direct' tension." *Num. Methods in Fracture Mech.*, A. R. Luxmoore et al., eds., Pineridge Press, Swansea, U.K., 457–471.
Rots, J. G., Kusters, G. M. A., and Blaauwendraad, J. (1987b). "Strain-softening simulations of mixed-mode concrete fracture." *Proc. SEM-RILEM Int. Conf. on Fracture of Concr. and Rock*, Houston, Tex., 226–240.
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