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SOMMARIO. Vengono esposti i risultati di una indagine teorica e sperimentale tendente a caratterizzare il comportamento meccanico e la resistenza a frattura di una trave fessurata a sezione circolare. Sebbene tali risultati non siano estrapolabili a geometrie più complesse, essi possono tuttavia risultare utili nella pratica, poichè la geometria circolare ricorre sia nel campo strutturale meccanico (alberi) che in quello civile (colonne, pilastri).

Il problema teorico viene risolto numericamente, utilizzando un modello discreto con elementi resistenti in serie e in parallelo, come usualmente si fa in elettrodinamica. Nella parte terminale dell'articolo, infine, i risultati di esperienze condotte su travi fessurate di PMMA (plexiglass) a sezione quadrata e circolare sono descritti e discussi alla luce del modello assunto.

SUMMARY. In the present paper, the results of a theoretical and experimental analysis are expounded, with the aim of characterizing the mechanical behaviour and the fracture resistance of a cracked beam with circular cross-section. Even if such results cannot be extrapolated to more complicated geometries, they can be of considerable practical importance, since the circular geometry recurs in the mechanical structural field (shafts) as well as in the civil one (pilasters, bearing piles). The theoretical problems is herein numerically solved, utilizing a discrete model with series and parallel resistant elements, as is usually made in electrodynamics. The results of tests carried out on cracked beams of PMMA with square and circular cross-sections are eventually shown and explained in the light of the assumed model.

1. INTRODUCTION

The presence of a crack in a structural element reduces the stiffness of the element and produces not only stress concentration, but also a redistribution of the stress field, which can be felt even at considerable distances. Regarding the stiffness variation of a beam due to a crack, fundamental contributions have been given by Liebowitz and Okamura [1 - 4] and by Herrmann [5]. An assumption of elastic hinge behaviour of a cracked beam cross-section has been formulated by Parmerter [6] to analyze a cracked beam on elastic Winkler foundation.

Considering a crack with free surfaces lying in a plane orthogonal to the beam axis (Fig. 1a) is generally a three-dimensional problem of Fracture Mechanics. This has been extensively studied by Hartranft and Sih [7, 8]. In fact, whereas in two-dimensional plane problems the crack tip is a geometric point and the crack itself is a curve (usually a segment), in the three-dimensional problems the crack is a discontinuity surface, bounded by the emerging crack on the external body surface, and by (or only by, if the crack is internal) the front, inside the body (Fig. 2).

Let us consider a point P_0 along the crack front (Fig. 2) and the local reference axes tnz, with origin P_0, coincident respectively with the tangent, normal and binormal to the front. The three stress-intensity factors are then defined in the following manner [9]:

$$
\begin{aligned}
K_I(P_0) &= \lim_{P \to P_0} (2\pi r)^{1/2} \sigma_z , \\
K_{II}(P_0) &= \lim_{P \to P_0} (2\pi r)^{1/2} \tau_{nz} , \\
K_{III}(P_0) &= \lim_{P \to P_0} (2\pi r)^{1/2} \tau_{tz} ,
\end{aligned} \tag{1}
$$

P being a point close to P_0, belonging to the normal plane nz, and r the distance of P from P_0 (Fig. 2). In each point of the crack front (Fig. 1a), the stress-intensity factors can be active all together, and each of them can vary along the crack front.

In the following section, a particular problem is solved where only the Mode I is taken into consideration. The global bending stiffness and the K_I-factor variation along the crack front are determined for a beam with a circular cross-section and a crack in the middle, by varying the beam length and the crack depth.

In order to verify the validity of the assumed discrete model consisting of in-series and in-parallel resistant elements, the results of an experimental analysis on polymethylmethacrylate (PMMA) beams with square and circular cross-sections are reported.

2. STIFFNESS LOSS OF A CRACKED BEAM

Consider a beam of brittle material, with rectangular cross-section, length $2l$, width b, thickness t, cracked in the middle and subjected to the bending moment M (Fig. 1b). Let:

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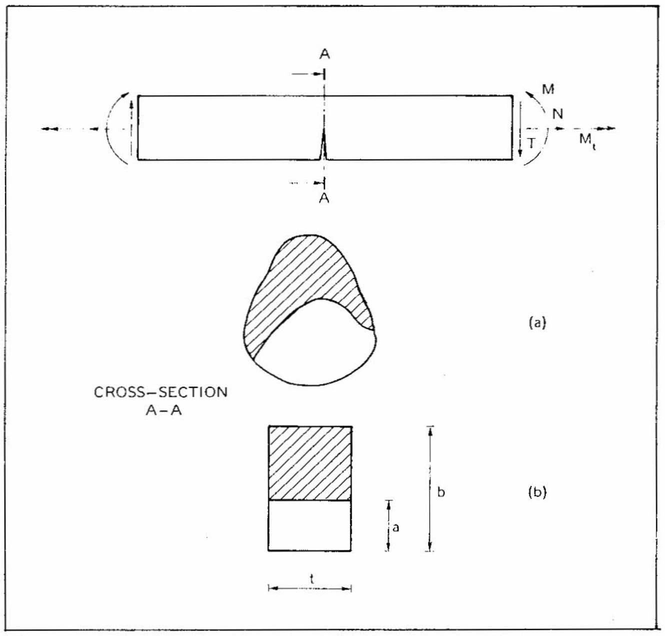

Fig. 1. Cracked beam element.

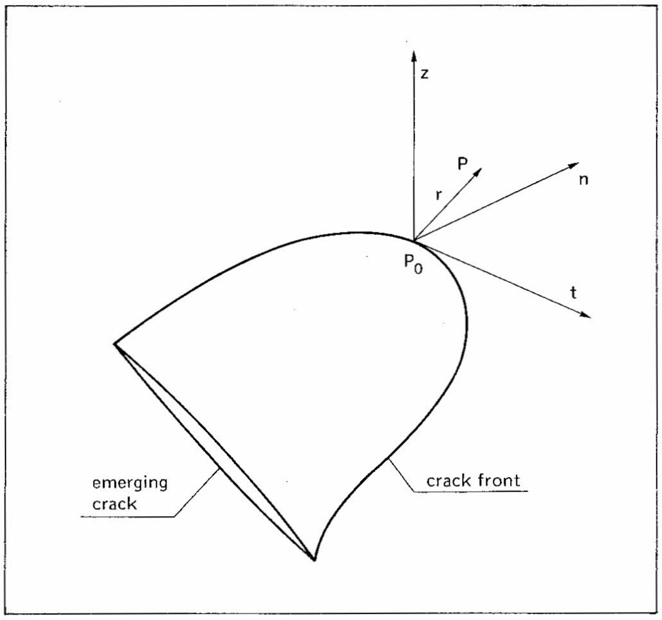

Fig. 2. Three-dimensional crack.

$$C = \frac{\vartheta}{M}, \tag{2}$$

be the local compliance of the cracked cross-section, ϑ being the relative rotation.

At first the remaining part of the beam will be considered rigid. The rotation ϑ increases as soon as the crack extends, and then the constant moment M performs positive work. The external work of the moment M is equal to the surface energy necessary to develop the new crack, plus the change in strain energy:

$$M\,\mathrm{d}\vartheta = \mathscr{G} t\,\mathrm{d}a + d\left(\frac{1}{2} M\vartheta\right), \tag{3}$$

where $\mathscr{G}$ is the generalized crack extension force and $\mathrm{d}\vartheta$ the rotation increment due to the elementary crack extension $\mathrm{d}a$. Applying (2) and the well-known relationship (material with Poisson ratio $\nu \to 0$):

$$\mathscr{G} = \frac{K_I^2}{E}, \tag{4}$$

where E denotes Young's modulus of the material, (3) becomes:

$$\frac{1}{2} M^2\,\mathrm{d}C = \frac{K_I^2}{E} t\,\mathrm{d}a. \tag{5}$$

If the factor K_I is expressed in the form [6]:

$$K_I = \frac{6M}{b^{3/2} t} g(\xi), \tag{6}$$

with $\xi = a/b$ = relative crack depth and [10]:

$$g(\xi) = \xi^{1/2}[2 - 2.435\,\xi + 10.19\,\xi^2 - 7.912\,\xi^3 + $$
$$+ 27.03\,\xi^6 - 16.51\,\xi^7],$$

for $0 \leqslant \xi \leqslant 0.5$, and:

$$g(\xi) = 0.663/(1-\xi)^{3/2}, \quad \text{for } 0.5 \leqslant \xi < 1, \tag{7}$$

then (5) becomes:

$$\mathrm{d}C = \frac{72 g^2(\xi)}{b^2 tE}\,\mathrm{d}\xi. \tag{8}$$

If both members are integrated between zero and the crack depth a, the local compliance of the cracked cross-section can be obtained:

$$C(\xi) = \frac{72}{b^2 tE} \int_0^{\xi} g^2(z)\,\mathrm{d}z = \frac{6 f(\xi)}{b^2 tE}, \tag{9}$$

with [6]:

$$f(\xi) = \xi^2[12 - 19.5\,\xi + 70.1\,\xi^2 - 97.6\,\xi^3 + 142\,\xi^4 - 138\,\xi^5 +$$
$$+ 128\,\xi^6 - 132\,\xi^7 + 379\,\xi^8 - 417\,\xi^9 + 131\,\xi^{10} + 313\,\xi^{12} -$$
$$- 357\,\xi^{13} + 102\,\xi^{14}],$$

for $0 \leqslant \xi \leqslant 0.5$, and:

$$f(\xi) = \frac{1.32}{(1-\xi)^2} - 1.78, \qquad \text{for } 0.5 \leqslant \xi < 1. \tag{10}$$

The remaining part of the beam can now also be considered as elastic. If the global compliance of the beam is to be determined, the distributed compliance must be added to the local one. Since the resistant elements are in series, the quantities to be summed are the compliances. If the cross-section is supposed to be square, $b = t = 2r$, the total stiffness will be:

$$S_{\text{tot}} = \frac{1}{C_{\text{tot}}} = \frac{1}{C_{2l} + C_{\text{crack}}} = \frac{S_{2l} \cdot S_{\text{crack}}}{S_{2l} + S_{\text{crack}}}, \tag{11}$$

where C_{2l} and S_{2l} denote respectively compliance and stiffness of the elastic beam of $2l$ length, and C_{crack} and S_{crack} compliance and stiffness of the cracked cross-section.

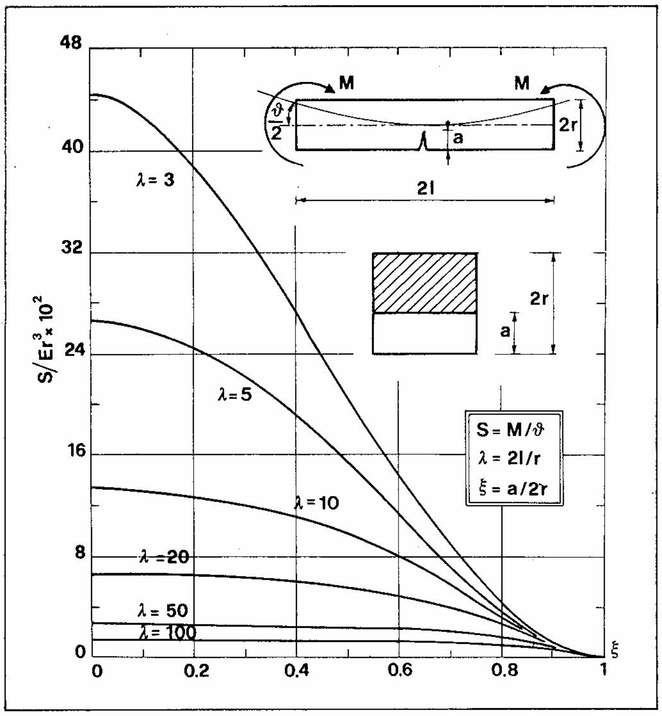

Fig. 3. Global bending stiffness of a cracked beam with square cross-section, as a function of crack depth ξ and beam slenderness λ.

Since:

$$S_{2l} = \frac{4}{3}\frac{r^4}{2l}E = Er^3\frac{4}{3\lambda}, \tag{12}$$

with $\lambda = 2l/r$ = beam slenderness, and from (9):

$$S_{crack} = \frac{b^2 tE}{6f(\xi)} = Er^3\frac{4}{3f(\xi)}, \tag{13}$$

the global stiffness is:

$$\frac{S_{tot}}{Er^3} = \frac{\frac{4}{3\lambda}\cdot\frac{4}{3f(\xi)}}{\frac{4}{3\lambda}+\frac{4}{3f(\xi)}} = \frac{4}{3[\lambda + f(\xi)]}. \tag{14}$$

In the diagram of Fig. 3 the stiffness (14) is represented as a function of the relative crack depth ξ and the beam slenderness λ. For $\xi = 0$, i.e. zero crack depth, the stiffness is the highest and is the same as the bending stiffness of the uncracked beam. For increasing ξ, the stiffness decays monotonically, taking the value zero for complete disconnection ($\xi = 1$). Then, for equal crack depths, the global stiffness decreases by increasing the slenderness λ. The diagram can be utilized even in the case of rectangular cross-section, of width $2r$ and thickness $t \neq 2r$; it is sufficient to multiply the values of the diagram by the ratio $t/2r$.

The case of a circular cross-section with rectilinear crack front is more complicated, because the resistant elements are not only in series, but partly *in series* and partly *in parallel*. Consider the cracked cross-section of Fig. 4, subjected to a bending moment with axis X. Such a cross-section can be subdivided into three sub-sections; each of them is related to a resistant beam element. While the external sub-sections A and C are the projections of an uncracked elastic beam, the central sub-section B is the projection of a cracked elastic beam. Such elements being in parallel, the sum of the stiffnesses of the elements A, B and C equals the global stiffness:

$$S_{tot} = S(A) + S(B) + S(C). \tag{15}$$

Since the elements A and C have the same stiffness and the element B consists of two in-series elements, i.e., the elastic beam of $2l$ length (cross-section B) and the cracked cross-section B, (15) becomes:

$$S_{tot} = 2S_{2l}(A) + \frac{S_{2l}(B)\cdot S_{crack}(B)}{S_{2l}(B) + S_{crack}(B)}. \tag{16}$$

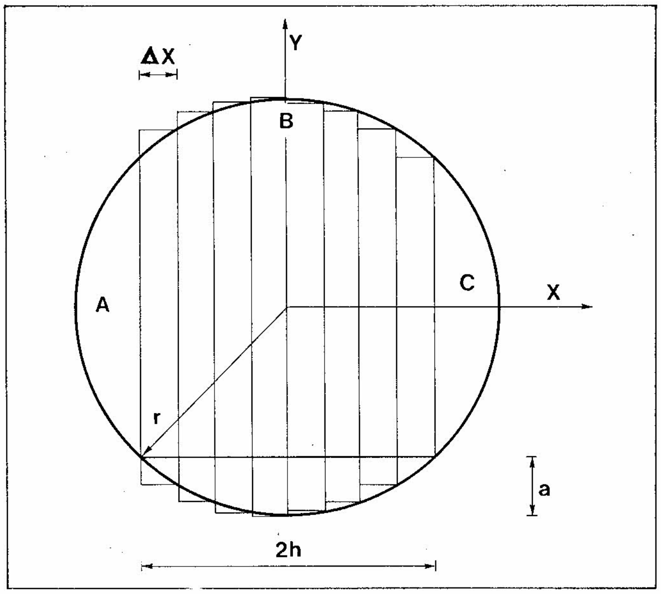

Fig. 4. Discretization of a circular beam cross-section with a crack of depth *a*.

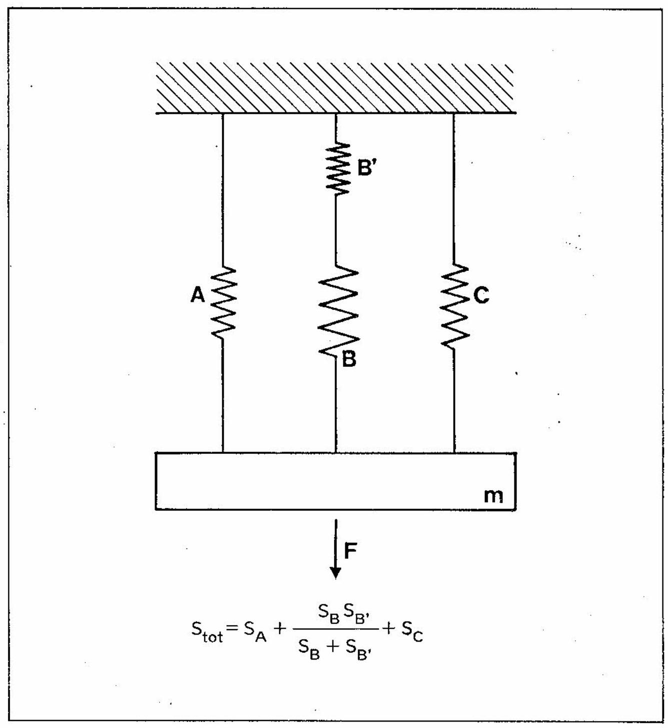

Fig. 5. Mechanical model with in-series and in-parallel springs.

In Fig. 5 a simplified mechanical model of the considered above elastic system is reported. Observe that, when the crack depth a is larger than the radius r of the cross-section, the external sub-sections A and C are the projections of an elastic beam completely disconnected, and the term $2\,S_{2l}(A)$ disappears in (16). The bending stiffness of the external elements is:

$$S_{2l}(A) = S_{2l}(C) = \frac{Er^4}{2l} I_A(\xi), \tag{17}$$

where ξ denotes the ratio between crack depth a and diameter $2r$, and:

$$I_A(\xi) = \frac{2}{3}\int_0^{\sin^{-1}(1-2\xi)} \sin^4\varphi \, d\varphi =$$

$$= \frac{1}{12}\,[3\varphi - 3\sin\varphi\cos\varphi - 2\sin^3\varphi\cos\varphi]_0^{\sin^{-1}(1-2\xi)}, \tag{18}$$

the non-dimensional moment of inertia of the external elements. Then the stiffness of the central elastic beam is:

$$S_{2l}(B) = \frac{Er^4}{2l} I_B(\xi), \tag{19}$$

with:

$$I_B(\xi) = \frac{\pi}{4} - 2I_A(\xi). \tag{20}$$

The circular cross-section provides a three-dimensional problem of Fracture Mechanics, since the stress-intensity factor K_I varies along the crack front. The stiffness $S_{\mathrm{crack}}(B)$ can be determined considering the cross-section B as constituted by N thin rectangular sub-sections (Fig. 4), of constant thickness:

$$\Delta X = \frac{2h}{N}, \tag{21}$$

where:

$$h = \sqrt{r^2 - (r-a)^2}, \tag{22}$$

and width:

$$b_i = 2\sqrt{r^2 - (-h + i\Delta X)^2}, \qquad i = 1, 2, \ldots, N. \tag{23}$$

A similar discrete model was utilized by Daoud, Cartwright and Carney [11] to compute the strain-energy release rate in a cracked circular bar subjected to axial force, and by Rice and Levy [12] in the analysis of an elastic plate with a part-through surface crack. Since the crack depth in the i^{th} rectangular sub-section is:

$$a_i = \frac{b_i}{2} - (r-a), \qquad i = 1, 2, \ldots, N, \tag{24}$$

the relative crack depth in the same sub-section is:

$$\xi_i = \frac{a_i}{b_i} = \frac{1}{2} - \frac{r-a}{b_i}, \qquad i = 1, 2, \ldots, N. \tag{25}$$

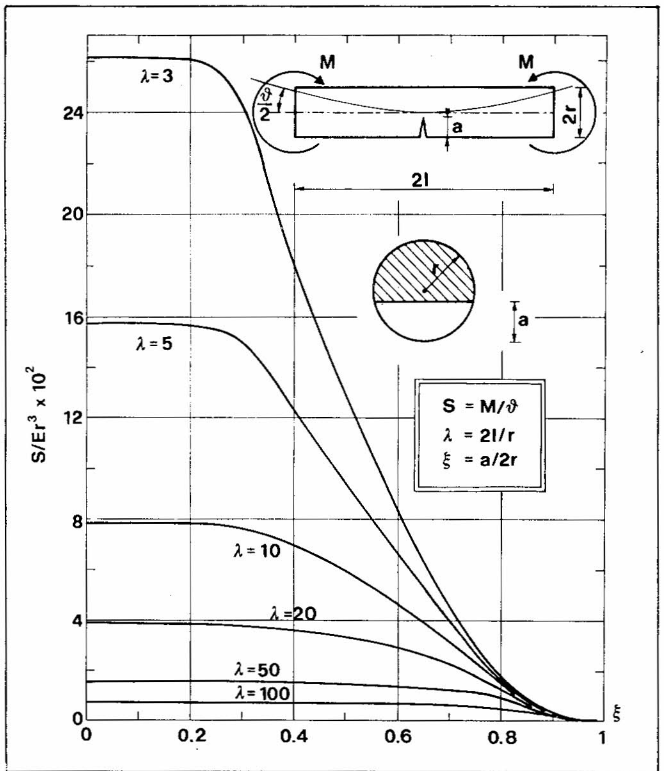

Fig. 6. Global bending stiffness of a cracked beam with circular cross-section, as a function of crack depth ξ and beam slenderness λ.

From (9) follows:

$$S_{\mathrm{crack}}(B) = \sum_{i=1}^{N-1} S_i = \sum_{i=1}^{N-1} \frac{b_i^2 \Delta X}{6 f(\xi_i)} E, \tag{26}$$

the thin rectangular elements being in parallel. The summation is extended up to $(N-1)$, since the N^{th} element has infinite stiffness. From (16), (17), (19) and (26), the global stiffness appears as follows:

$$\frac{S_{\mathrm{tot}}}{Er^3} = 2\,\frac{I_A(\xi)}{\lambda} + \frac{\dfrac{I_B(\xi)}{\lambda} * \dfrac{(\Delta X/r)}{6} \displaystyle\sum_{i=1}^{N-1} \frac{(b_i/r)^2}{f(\xi_i)}}{\dfrac{I_B(\xi)}{\lambda} + \dfrac{(\Delta X/r)}{6} \displaystyle\sum_{i=1}^{N-1} \frac{(b_i/r)^2}{f(\xi_i)}}, \tag{27}$$

for $0 \leqslant a \leqslant r$, and:

$$\frac{S_{\mathrm{tot}}}{Er^3} = \frac{\dfrac{I_B(\xi)}{\lambda} * \dfrac{(\Delta X/r)}{6} \displaystyle\sum_{i=1}^{N-1} \frac{(b_i/r)^2}{f(\xi_i)}}{\dfrac{I_B(\xi)}{\lambda} + \dfrac{(\Delta X/r)}{6} \displaystyle\sum_{i=1}^{N-1} \frac{(b_i/r)^2}{f(\xi_i)}}, \tag{28}$$

for $r \leqslant a < 2r$.

An analogous algorithm can naturally be organized to compute the bending stiffness of a beam with generic cross-section and a crack with curvilinear front, as has been suggested by Parmerter [6]. From (27) and (28) the stiffness appears to be, as in the case of square cross-section, a function of the beam length as well as of the crack depth. In the diagram of Fig. 6 such a stiffness has been reported as a function of ξ and λ. The computation has been carried out for $N = 1000$; for such a value it is already in asymptotic zone. In this case too, the curves have the meaning and the trends discussed in the case of square cross-section (Fig. 3). The only substantial difference between the two diagrams is that, in the case of circular cross-section and for low values of ξ, the curve presents a plateau. Naturally it is due to the particular geometry of the problem and, more precisely, to the fact that the crack-free surface does not increase linearly with the crack depth a.

The final purpose of this section is to study the ratio between the bending stiffness of a cracked beam element (rectangular cross-section) and the bending stiffness of a beam element with the cross-section equal to the ligament (Fig. 7). The stiffness of the element of length $2l$ and width $(2r - a)$ is:

$$\frac{S_0}{Er^3} = \frac{4}{3\lambda}(1-\xi)^3, \tag{29}$$

while the stiffness S of the cracked element of length $2l$ and width $2r$ is provided by (14). Then the ratio is:

$$\frac{S}{S_0} = \frac{\lambda}{[\lambda + f(\xi)](1-\xi)^3}. \tag{30}$$

This ratio is represented in Fig. 7, as a function of ξ and λ. While for sufficiently long elements the stiffness of the cracked element is definitely prevalent and for sufficiently short elements the stiffness of the uncracked element is, for medium lengths ($\lambda \simeq 0.4$) a sinusoidal variation of the curve occurs. At first the stiffness of the cracked element prevails, then that of the uncracked one, and eventually, for deep cracks, the prevalence of the stiffness of the cracked element occurs.

3. FRACTURE RESISTANCE OF A CRACKED BEAM

The purpose of the present section is to determine the

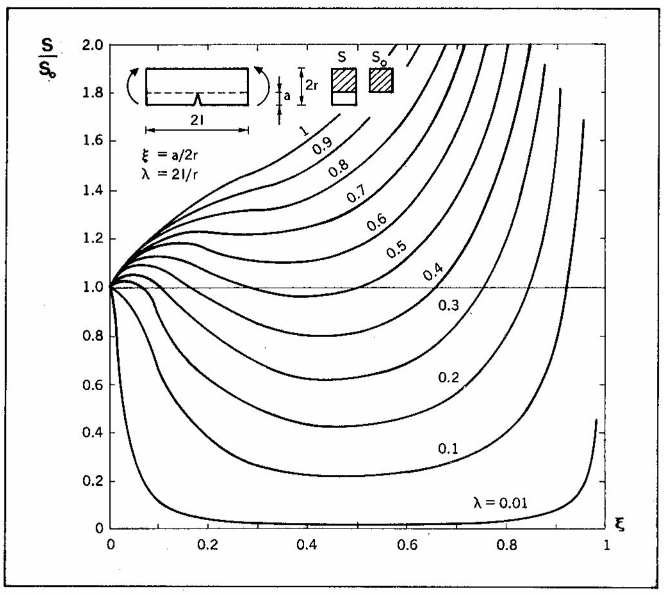

Fig. 7. Ratio between the stiffness of a cracked beam element and the stiffness of a beam element with the cross-section equal to the ligament.

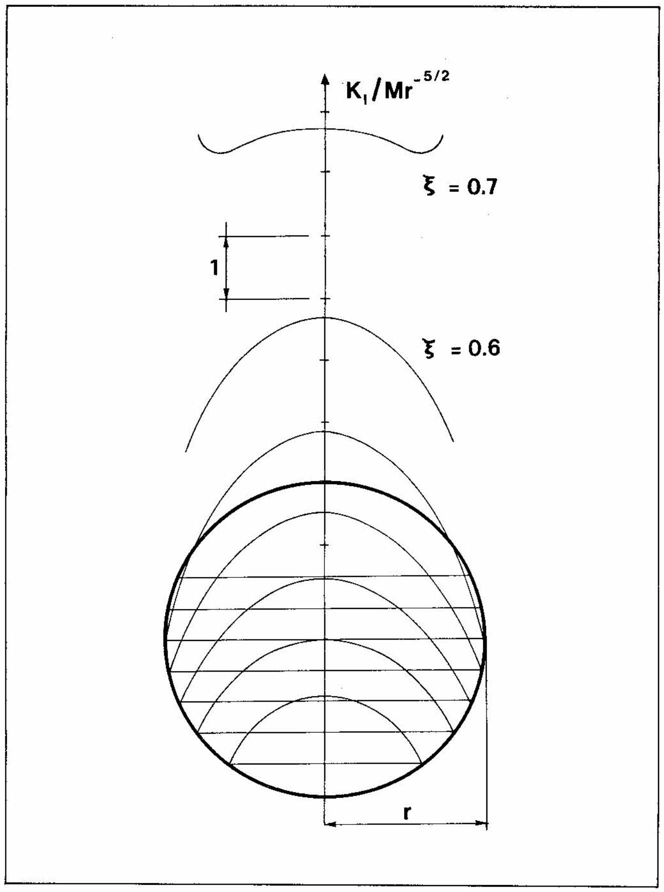

Fig. 8. Variation of the stress-intensity factor along the crack front, varying the crack depth.

stress-intensity factor K_I in each point of the crack front for a beam with circular cross-section. The finite element model of the preceding section will be adopted again and a solution independent of the width ΔX of the strips will be determined.

In order to attribute a fraction M_i of the external moment M to every strip, a condition of statical equivalence and one of geometrical compatibility will be used:

$$M = \sum_{i=1}^{N} M_i + M_A + M_C; \qquad \vartheta_i = \text{constant}. \tag{31}$$

If the beam is considered of infinite length, (31) provides:

$$M_i = M \frac{1}{3\pi} \frac{\Delta X b_i^3}{r^4}. \tag{32}$$

The application of (6) gives:

$$K_{I_i} = \frac{6M_i}{b_i^{3/2}\Delta X} g(\xi_i) = \tag{33}$$

$$= \frac{2}{\pi} M \frac{b_i^{3/2}}{r^4} g(\xi_i),$$

where b_i and ξ_i are given by (23) and (25). Note that the width ΔX of the strips has disappeared in (33). Then the sequence (33) can be replaced by a function defined for all real ξ:

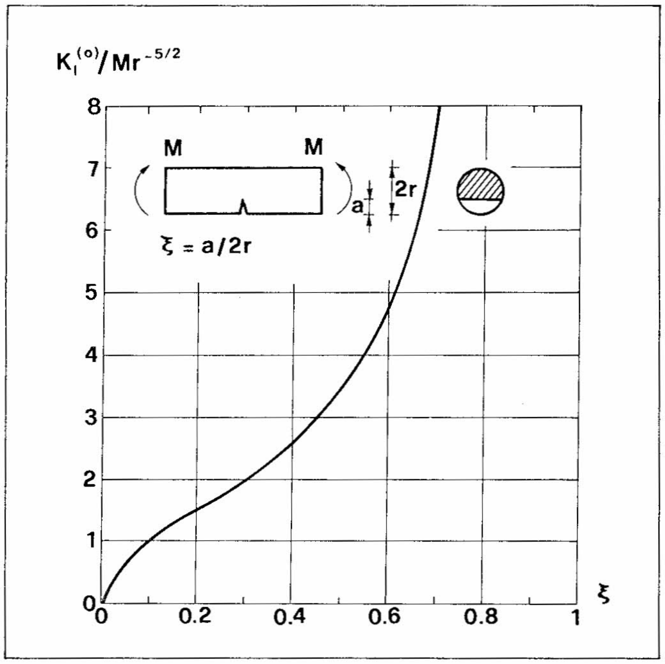

Fig. 9. Stress-intensity factor at the centre of the crack front, as a function of the crack depth.

$$\frac{K_I}{Mr^{-5/2}} = \frac{2}{\pi}\left(\frac{b}{r}\right)^{3/2} g(\xi). \tag{34}$$

In Fig. 8 the variation of the factor K_I along the crack front is plotted for several values of crack depth. In Fig. 9 the factor $K_I^{(0)}$, at the centre of the crack front, is plotted as a function of crack depth.

4. EXPERIMENTAL RESULTS AND DISCUSSION

In order to verify the validity of the discrete model assumed in the preceding sections, two series of tests have been carried out on PMMA three-point bent beams with square and circular cross-sections. In all, 48 beams, with side or diameter $2r$ = 5 cm and length $2l$ = 40 cm, have been tested. The crack was made using a circular saw of 0.5 mm thickness,

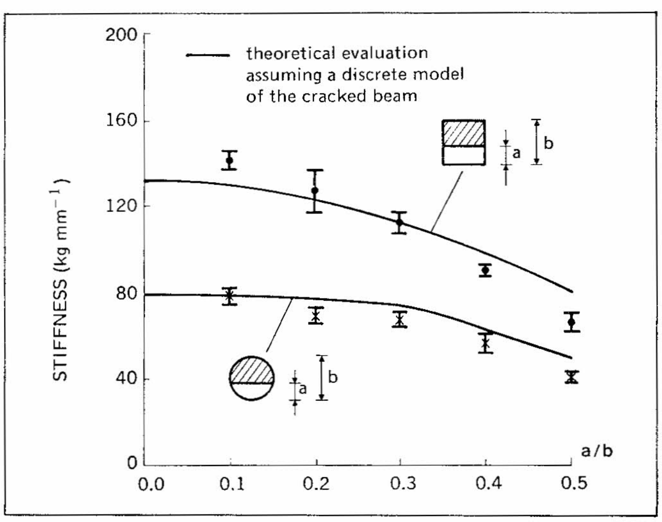

Fig. 10. Stiffness, relating to the vertical displacement at the beam centre: theoretical values, obtained by a discrete model, and experimental values, obtained by the three point bending tests carried out by the author.

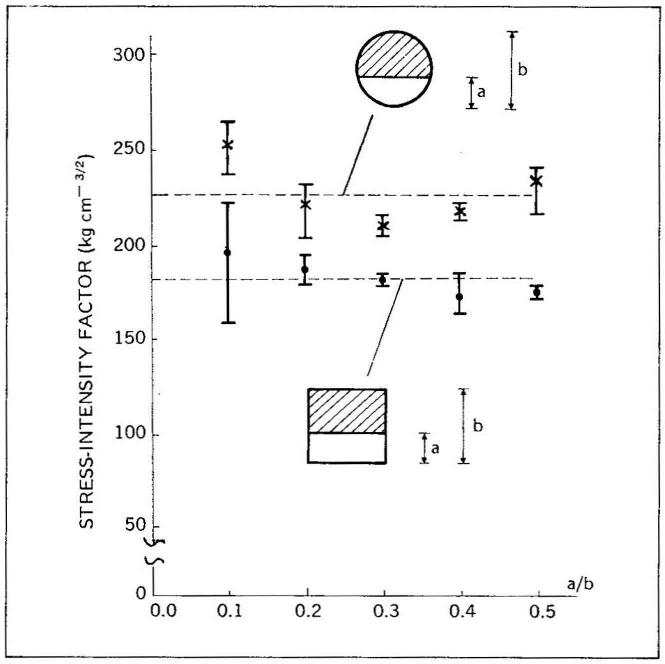

Fig. 11. Experimental values of fracture toughness K_{IC} obtained by PMMA beams.

and the crack depth a/b was chosen to be 0.0, 0.1, 0.2, 0.3, 0.4 and 0.5. Four identical specimens for each crack depth and for each geometry were tested.

The elastic Young's modulus has been obtained from tests on uncracked beams (E = 31,600 kg cm^{-2}).

In Fig. 10, the theoretical stiffness, computed by the discrete model, is reported (continuous line) together with the experimental results.

The experimental results confirm the existence of a plateau and a knee in the case of the circular geometry. In deed, the stiffness is nearly constant for $a/b \lesssim 0.3$ and rapidly decreasing for higher values of a/b. The crack depth for which the knee occurs ($a/b \simeq 0.3$) is coincident with the crack depth for which the stress-intensity factor at the center of the crack front presents a deflection point (Fig. 9). Therefore, cracks with depths $a/b \gtrsim 0.3$ must be considered particularly dangerous in bent beams with circular cross-sections.

The computation of the critical factor K_{IC} (fracture toughness) has been carried out in a way similar to that of section 3, with the difference that the length of the beams utilized is not infinite. That is, the load fraction acting on each strip cannot be evaluated simply in proportion to the moment of inertia, but must also account for the crack effect and then be proportional to the stiffness of the ith element, i.e., of a beam of length $2l$ with rectangular cross-section of width b_i, thickness ΔX and a crack of depth ξ_i.

In Fig. 11, the experimental results are reported. Whereas the mean value of the fracture toughness for the square beams is 182.35 kg cm$^{-3/2}$, for the circular ones it works out to be 227.68 kg cm $^{-3/2}$. This discordance proves that the real maximum of K_I along the crack front is lower than the computed one, and that the external parts of the section carry a fraction of load slightly larger than the one predicted by the assumed discrete model.

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