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Buckling instability of a von Koch beam

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A B S T R A C T

In this paper, we commemorate the centenary since death of the Swedish mathematician Niels von Koch (and the 120th anniversary of the birth of its famous fractal set). The buckling analysis of a von Koch beam is investigated, which can effectively define the elastic stability problem for fractal-shaped antennas or more general and random natural forms. Results depending on the fractal dimension of the structure suggest peculiar scaling laws for the buckling load of fractal-shaped beams. It is found that the buckling load tends to zero when the iteration $n \to \infty$, proving the high sensitivity to buckling of fractal antennas and trees. As in the case of free vibration, the eigenvalue tends to zero: the buckling load in the present case as well as the resonance frequency in dynamics. In the case of vibration, the fractal effect is extremely beneficial, whereas it is extremely dangerous in the case of buckling instability. The sudden collapse of a natural tree after a wind gust can be produced by elastic buckling much more likely than by dynamic resonance.

1. Introduction: fractals and complex patterns

Mandelbrot (1982) observed in the natural world a series of irregularities, tortuosities, and discontinuities, which cannot be described with classical mathematics. In this sense, he opposed Leibniz (Leibniz, 1765), the philosopher who lived at the turn of the 17th and 18th Centuries and was the founder of infinitesimal calculus, whose well-known apothegm states "Natura non facit saltus", i.e., "Nature makes no leap". Two worlds therefore collide, representing two ways of thinking: on the one hand the optimism, firstly related to the age of Enlightenment, and then to Positivism, on the other the skepticism of critical rationalism, perhaps disillusioned, but also projected on new and exciting paradigms (Popper, 1934). Pythagoras (VI Century BC) already taught us that nature, with its irregularities and anomalies, can be described by mathematics: "The numbers are the beginning of all things. The whole universe is harmony and number" (Carpinteri, 1998, 1999). Pythagoras gives us a first significant example of how to obtain complex forms from elementary geometries, by recursive procedures. The application of his well-known theorem, repeated infinitely, gives a logarithmic spiral. Like mathematics, with its abstract recursive procedures, produces extremely complex geometric figures, similarly physics, chemistry, and biology, with recursive procedures – in this case real – create mineral crystals, plants, or animals even more complex (e.g., spiral shells, shrubs, flowers, ferns, trees, forests, etc.).

Even the artists try to reproduce the complexity of nature, with procedures hitherto considered irrational, but certainly also of a recursive type at the moment of inspiration. The brushstrokes of the Impressionist painters overlap sometimes in a systematic way, and sometimes only apparently disorderly. The notes of the most famous musicians are chasing each other without apparent rules, but in a harmonious way. Moreover, the chemical phenomena of growth by diffusion and aggregation create branched structures of dendritic type. Classical examples are electrolytic deposition, diffusion of fluids into other fluids, electrical discharges, and bacterial colonies. Urban aggregations also form and develop with similar rules and morphological results. So much that the Argentinian writer Borges rightly defines Buenos Aires in its years of great expansion as "a city growing like a tree" (Carpinteri, 1998).

In the animal world too, it is common to encounter structures or systems branched recursively. As an example, in human body we find the blood vessel system, or bronchi-system, as well as the nervous system. In addition, each neuron can be considered as a complex branched system, proving that there is self-similarity between the whole nervous system and its parts.

Furthermore, the fractures that originate and grow in metals, rocks, concrete, often have a branched and self-similar shape (Carpinteri, 1994a, 2021). This means that peculiar geometric features are present at any length-scales, so that it may become difficult to distinguish between the morphology of a seismic fault and that of a corrosion microcrack.

The mountain contours themselves have sometimes complex and self-similar morphologies, so that often, without reference objects, it becomes difficult to guess the scale of the image: it could be a dolomite peak, but also a detail of a trivial stone.

Properties of self-similarity, which is the repetition of the same

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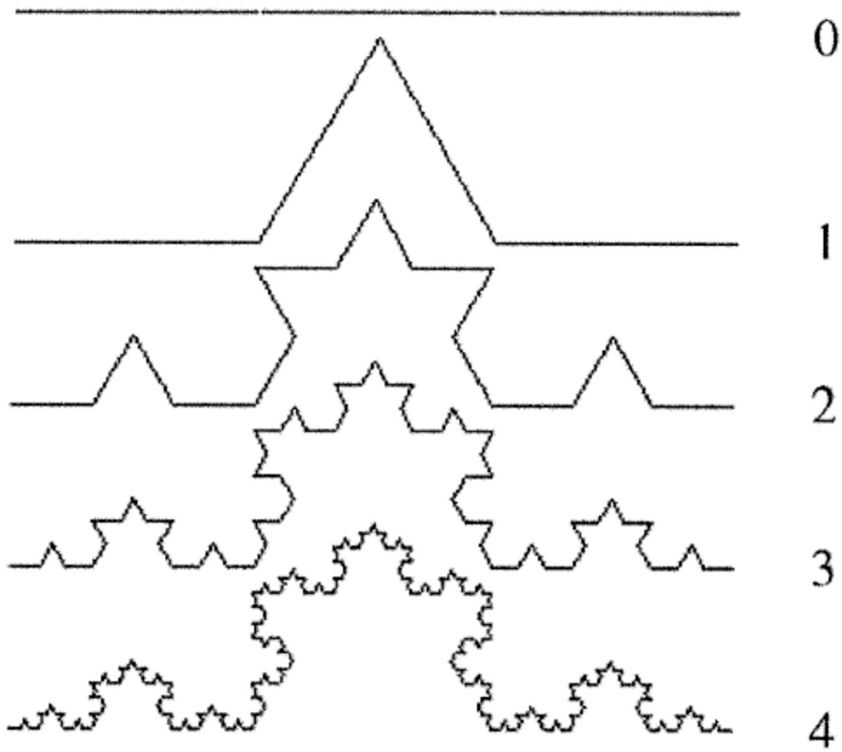

Fig. 1. Triadic von Koch curve (4 iterations).

geometric or statistic characteristics at any length-scale, are therefore often present in nature, from the microscopic scale (trajectories of the Brownian motions of elementary particles) to the intergalactic one (stars, galaxies, and clusters of galaxies). Self-similarity also concerns temporal phenomena. The graphs of heart rate, stock index, or earthquake magnitude, when plotted as a function of time, resemble each other on any time-scale.

The dynamic behaviour of chaotic systems is unpredictable, since a small variation in the causes can lead to a huge variation in the effects. Determinism and classical mechanics thinking are thus brought into crisis when a multiplicity of solutions is presented for the same problem. The "butterfly effect" by Lorenz (1972) is the first example of this type of problems, describing the unpredictability of weather conditions. Therefore we have, on the one hand, chaotic dynamics with its indeterminism, on the other hand, classical Newtonian mechanics with its determinism, which is capable of predicting the oscillation of a pendulum or the orbit of a planet with millimetric precision. There is no complete separation between these two worlds. Even the astral movements appear chaotic, at very large spatial and temporal scales. In the same way, atomic phenomena appear chaotic and undetermined at equally unusual scales.

In this context, the advent of Fractal Geometry has improved the understanding of processes of natural formation or transformation, offering new algorithms for extrapolating these processes (Mandelbrot, 1967). Fractal mathematics has produced new means of expression that have been soon recognized as useful for the advancement of several scientific disciplines.

The term "fractal", which comes from the Latin "fractus", i.e., "fragmented", was introduced by Mandelbrot (1975) to indicate a system that shows the property of self-similarity, such that by enlarging any part, however small, of the system, it has a structure identical to that of the whole system. In general, such property returns a dimension that is not integer. Mandelbrot has shown with numerous and suggestive examples that concepts considered as abstract curiosities constitute instead a new type of mathematical apparatus for the description of intrinsically irregular structures (Carpinteri, 1994a,b; Falconer, 2014; Mandelbrot, 1967, 1982; Panagiotopoulos et al., 1993; Rian et al., 2018; Waller, 2006).

More in particular, invasive fractals are sets having a dimension greater than that of reference set. Their archetype is represented by the von Koch curve (von Koch, 1904, 1906). It can be constructed from an interval through an infinite sequence of operations of substitution of the "middle third" with the other two sides of the equilateral triangle that has its base on the removed segment (Fig. 1). The length, in classical terms, of the von Koch curve is infinite. It is possible to show, however, that the fractal dimension of this set is the noninteger number 1.262 and that the curve therefore presents a finite measure only in relation to a unit of length raised to such an anomalous exponent.

2. The triadic von Koch curve

In 2024 we commemorate the centenary of the death of the Swedish mathematician Niels Fabian Helge von Koch (1870–1924). We also celebrate the 120th anniversary of the birth of its famous fractal curve (von Koch, 1904). This fractal set represents one of the most emblematic geometric figures of the 20th Century and an inspiration to many scientific works, such as that of Mandelbrot, who devoted an entire chapter of his fundamental book to it (Mandelbrot, 1982). In particular, the potentiality of von Koch curve for improving the efficiency of natural or man-made objects has recently been suggested in different research fields (Carpinteri et al., 2009, 2010; Lakes, 1995). Fractal-shaped antennas or more general natural objects have some unique characteristics that are linked to the geometrical properties of von Koch curve.

Let us recall the properties of the triadic von Koch curve (Fig. 1), V, which splits into four parts V_j (j = 1, 2, 3, 4) geometrically similar to V but scaled by the factor 1/3:

$$V = \bigcup_{j=1}^{4} V_j \tag{1}$$

The heuristic definition of the Hausdorff dimension returns:

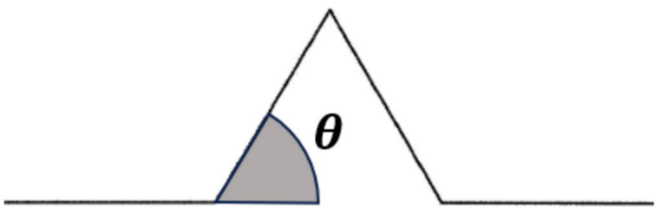

Fig. 2. Indentation angle θ.

$$Mes(V) = 4Mes(V_j) = 4\left(\frac{1}{3}\right)^{\alpha} Mes(V) \tag{2}$$

where α is the fractal dimension of the set by definition.

If we divide both sides by the Hausdorff measure, we obtain:

$$1 = 4\left(\frac{1}{3}\right)^{\alpha} \tag{3}$$

Thus:

$$\alpha = \frac{\log(4)}{\log(3)} \approx 1.262 \tag{4}$$

The result reported above refers to the traditional von Koch curve, where the indentation angle is $\theta = 60^\circ$ (Fig. 2).

Considering different indentation angles (Fig. 3), an analogous

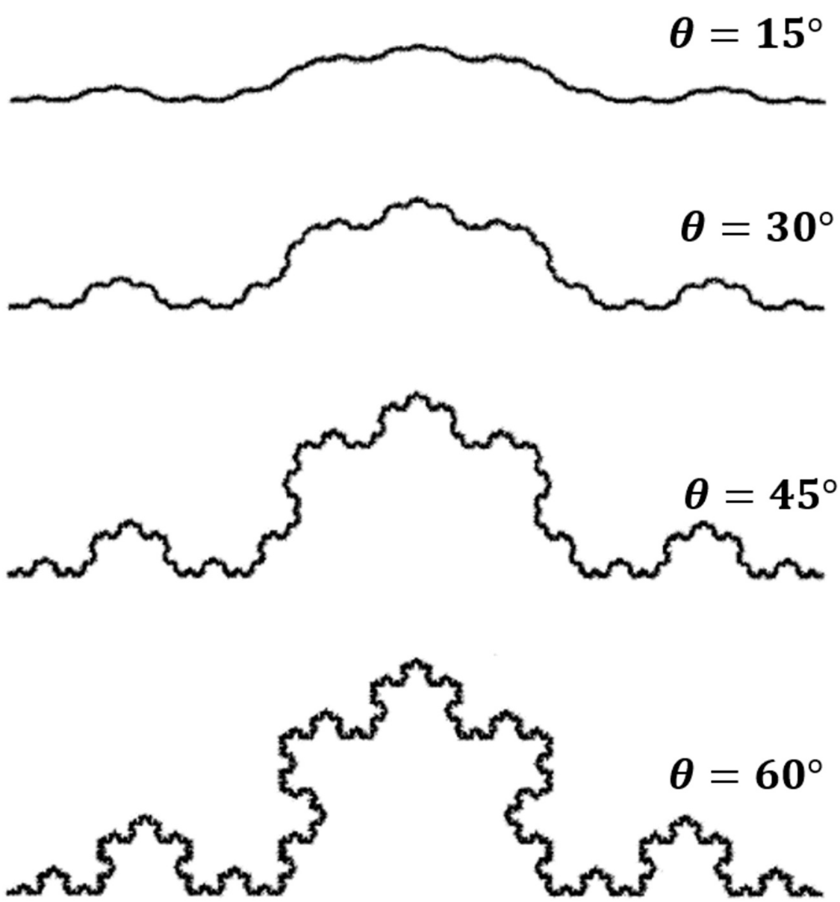

Fig. 3. von Koch curves (after 5 iterations) with indentation angles $\theta = 15^\circ, 30^\circ, 45^\circ, 60^\circ$.

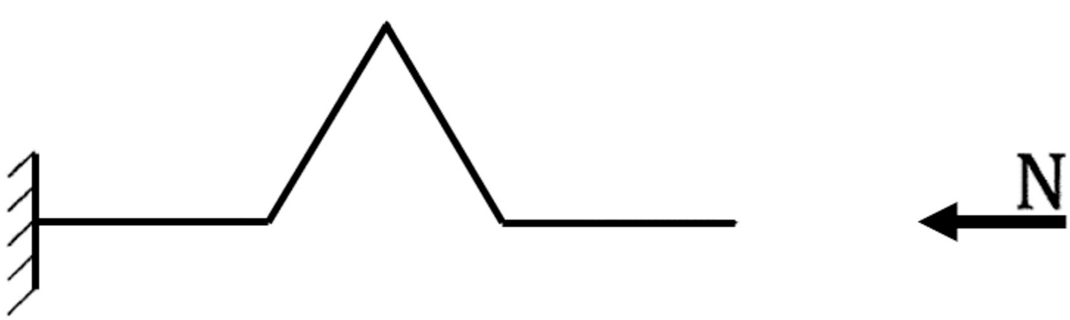

Fig. 4. von Koch cantilever beam after 1 iteration.

Table 1
Critical multipliers of the load.

Iteration, n	$N_{cr,n}$ ($\theta = 15^\circ$)	$N_{cr,n}$ ($\theta = 30^\circ$)	$N_{cr,n}$ ($\theta = 45^\circ$)	$N_{cr,n}$ ($\theta = 60^\circ$)
0	2.47	2.47	2.47	2.47
1	2.42	2.30	2.10	1.84
2	2.38	2.15	1.79	1.38
3	2.34	2.00	1.53	1.03
4	2.30	1.87	1.31	0.78

balance may be written:

$$1 = 2\left(\frac{1}{3}\right)^{\alpha} + 2\left(\frac{1}{6\cos\theta}\right)^{\alpha} \approx 4\left(\frac{1}{2(1+\cos\theta)}\right)^{\alpha} = 4(q)^{\alpha} \quad (5)$$

Then:

$$\alpha = \frac{\log(4)}{\log(q^{-1})} \quad (6)$$

3. Buckling instability

Let us now consider the buckling instability analysis of a triadic von Koch cantilever beam (Fig. 4). The buckling condition ($N_{cr} = \frac{\pi^2 EI}{4L_0^2}$, only for the initiator) is evaluated for 4 iterations (Fig. 1) and for 4 different indentation angles θ (Fig. 3), considering the following geometrical and mechanical characteristics: $I = 1000$ mm^4; $L_0 = 1000$ mm (rectilinear beam); $E = 1000$ N/mm^2. The following eigenvalue equation is solved for each framed beam system of the pre-fractal sequence (Carpinteri et al., 2009, 2010):

$$\det([K] - N[K_g]) = 0 \quad (7)$$

where $[K]$ and $[K_g]$ are the elastic and geometric stiffness matrix, respectively, of each pre-fractal framed beam system. In particular, for the generic i-th beam element, the elastic stiffness matrix may be cast in the form:

$$[K_i] = \int_0^{l_i} EI_i\{\eta_i''\}\{\eta_i''\}^{\mathrm{T}}\mathrm{d}z \quad (8)$$

where
l_i represents the length of the i-th beam element;
I_i represents the inertia of the i-th beam element cross-section;
$\{\eta_i\}$ denotes the shape function vector.

On the other hand, the geometrical stiffness matrix of the i-th beam element can be cast in the form:

$$[K_{gi}] = \int_0^{l_i} \{\eta_i'\}\{\eta_i'\}^{\mathrm{T}}\mathrm{d}z \quad (9)$$

To solve the problem, the subsequent operations involve rotation and expansion of the local stiffness matrices (Carpinteri, 2017). Finally, the assemblage operation provides the global stiffness matrices, so that the eigenvalue problem can be formulated following Eq. (7). Then, the solution for the fractal is found as the limit of the solutions obtained for the pre-fractals.

The minimum eigenvalue N is said to be the critical multiplier of the loads. Concerning the problem of stability of the elastic equilibrium, the critical multiplier represents the load of incipient collapse and it multiplies the unit force (1 Newton). The results of the buckling instability

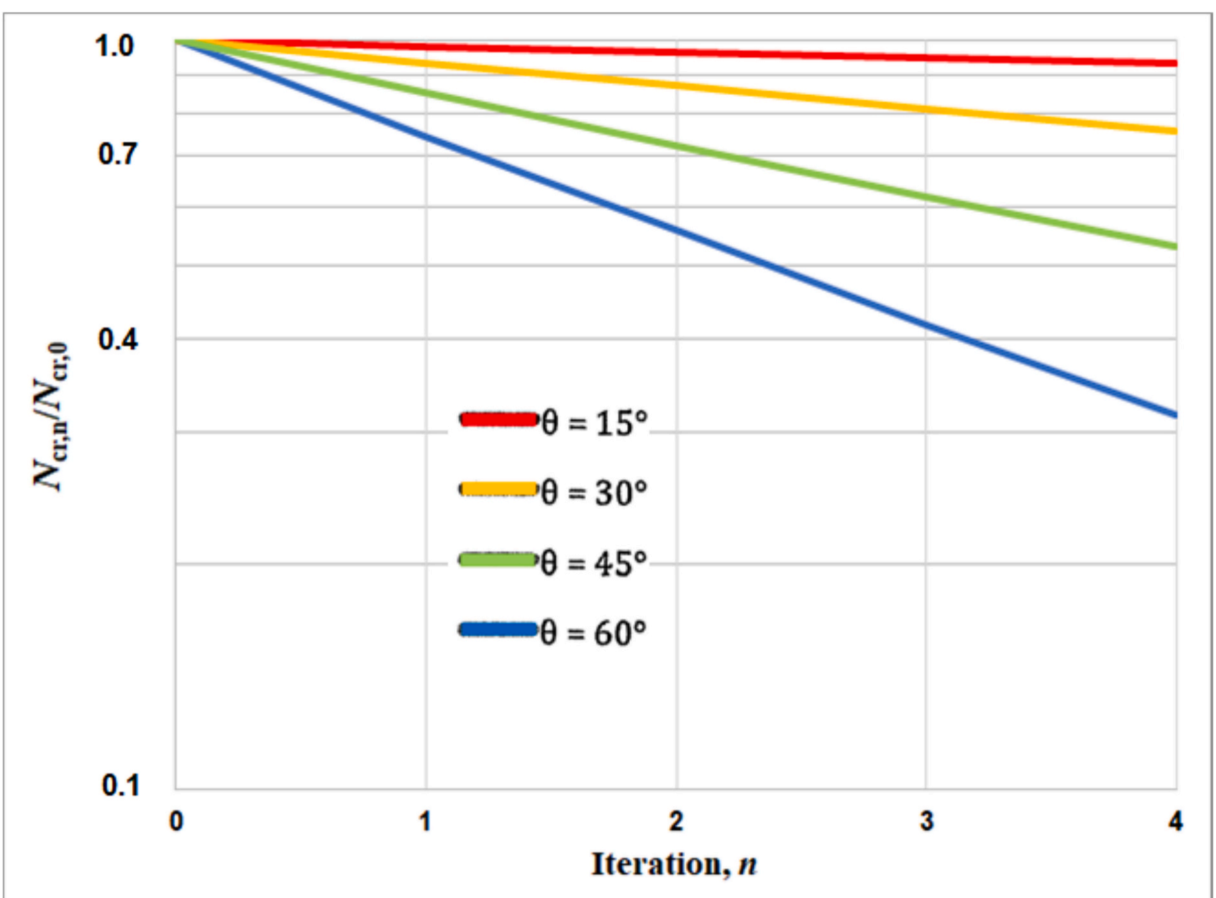

Fig. 5. Fractal buckling load.

analysis are reported in Table 1 by varying θ and increasing n, showing that the critical multiplier tends to zero when $n \to \infty$ (extremely negative fractal effect).

The results shown in Table 1 are graphically represented in Fig. 5, where the normalized buckling load $\log(N_{cr,n}/N_{cr,0})$ is displayed for each indentation angle θ and iteration n. The slope of the curves in Fig. 5 is equal to log(4q).

4. Conclusions

The buckling analysis of a von Koch cantilever beam, which can effectively define the elastic stability problem of fractal-shaped antennas or trees, is investigated in this paper. Results depending on the fractal dimension of the structure suggest peculiar scaling laws for the buckling load of the fractal antennas or trees. In particular, simple recursive relationships emerge, as also evidenced in the case of free vibration of the von Koch beam (Carpinteri et al., 2009, 2010). Eventually, it is found that the buckling load tends to zero when $n \to \infty$, proving the high sensitivity to buckling of fractal antennas and trees. As in the case of free vibration, the eigenvalue tends to zero: the buckling load in the present case as well as the resonance frequency in dynamics. In the case of vibration, the fractal effect is extremely beneficial, whereas it is extremely dangerous in the case of buckling instability. As an original remark, we can conclude that the sudden collapse of a natural tree after a wind gust can be produced by elastic buckling much more likely than by dynamic resonance.

CRediT authorship contribution statement

Alberto Carpinteri: Writing – review & editing, Validation, Supervision, Project administration, Conceptualization. **Federico Accornero:** Writing – original draft, Investigation, Formal analysis, Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

References

Carpinteri, A., 1994a. Fractal nature of material microstructure and size effects on apparent mechanical properties. Mech. Mater. 18, 89–101.
Carpinteri, A., 1994b. Scaling laws and renormalization groups for strength and toughness of disordered materials. Int. J. Solids Struct. 31, 291–302.
Carpinteri, A., 1998. Frattali, catastrofi e leggi di scala nella meccanica dei materiali. Quaderni dell'Accademia delle Scienze di Torino 7, 17–36.
Carpinteri, A., 1999. La Geometria della Natura non é Euclidea: Gerarchia, Disordine, Frattalitá. Edizioni dell'Accademia Teodoriana, Capua.
Carpinteri, A., 2017. Advanced Structural Mechanics. CRC Press, New York.
Carpinteri, A., 2021. Fracture and Complexity. Springer Nature, Heidelberg.
Carpinteri, A., Pugno, N., Sapora, A., 2009. Asymptotic analysis of a von Koch beam. Chaos, Solitons, and Fractals 41, 795–802.
Carpinteri, A., Pugno, N., Sapora, A., 2010. Free vibration analysis of a von Koch beam. Int. J. Solids Struct. 47, 1555–1562.
Falconer, K.J., 2014. Fractal Geometry: Mathematical Foundations and Applications. Wiley, New York.
Lakes, R., 1995. Materials with structural hierarchy. Nature 361, 511–515.
Leibniz, G.W., 1765. Oeuvres Philosophiques, Raspe, Amsterdam.
Lorenz, E., 1972. Predictability: Does the flap of a butterfly's wings in Brazil set off a tornado in Texas? *Presented before the American Association for the Advancement of Science (AAAS), December 29, 1972*, Washington DC.
Mandelbrot, B., 1967. How long is the coast of Britain? Statistical Self-Similarity and Fractional Dimension. Science 156, 636–638.
Mandelbrot, B., 1975. *Les Object Fractals: Forme*. Hasard et Dimension, Flammarion, Paris.
Mandelbrot, B., 1982. The Fractal Geometry of Nature. Freeman, San Francisco.
Panagiotopoulos, P.D., Panagouli, O.K., Mistakidis, E.S., 1993. Fractal geometry and fractal materials in solids and structures. Arch. Appl. Mech. 63, 1–24.
Popper, K.V., 1934. The Logic of Scientific Discovery. Hutchinson, New York.
Rian, I.M., Sassone, M., Asayama, S., 2018. From fractal geometry to architecture: Designing a grid-shell-like structure using the Takagi-Landsberg surface. Computer-Aided Design 98, 40–53.
von Koch, N.F.H., 1904. Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire. Arkiv För Matematik, Astronomi Och Fysik 1, 681–702.
von Koch, N.F.H., 1906. Une méthode géométrique élémentaire pour l'étude de certaines questions de la théorie des courbes planes. Acta Mathematica 30, 145–174.
Waller, S.D., 2006. Mechanics of Novel Compression Structures. University of Cambridge. PhD dissertation.