Research Article

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Funicularity in elastic domes: Coupled effects of shape and thickness**

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Abstract: An historical overview is presented concerning the theory of shell structures and thin domes. Early conjectures proposed, among others, by French, German, and Russian Authors are discussed. Static and kinematic matrix operator equations are formulated explicitly in the case of shells of revolution and thin domes. It is realized how the static and kinematic matrix operators are one the adjoint of the other, and, on the other hand, it can be rigorously demonstrated through the definition of stiffness matrix and the application of virtual work principle. In this context, any possible omission present in the previous approaches becomes evident. As regards thin shells of revolution (thin domes), the elastic problem results to be internally statically-determinate, in analogy to the case of curved beams, being characterized by a system of two equilibrium equations in two unknowns. Thus, the elastic solution can be obtained just based on the equilibrium equations and independently of the shape of the membrane itself. The same cannot be affirmed for the unidimensional elements without flexural stiffness (ropes). Generally speaking, the static problem of elastic domes is governed by two parameters, the constraint reactions being assumed to be tangential to meridians at the dome edges: the shallowness ratio and the thickness of the dome. On the other hand, when the dome thickness tends to zero, the funicularity emerges and prevails, independently of the shallowness ratio or the shape of the dome. When the thickness is finite, an optimal shape is demonstrated to exist, which minimizes the flexural regime if compared to the membrane one.

Keywords: static-kinematic duality, shell of revolution, thin domes, membranes, shallowness ratio, dome thickness, funicularity

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1 Introduction

Theory of space structures was limited to the study of plane trussed frameworks until the last decades of the 19th Century [1, 2]. Johann Wilhelm Schwedler (1823-1894), a German engineer and chief of the Royal Prussian Railways (Figure 1), improved the design of domes providing not only an early theoretical approach, but also a simplified structural calculation technique [2, 3]. In 1875 he designed the roof of a gasometer for the Imperial Continental Gas Association in Berlin-Kreuzberg, an iron dome structure with a diameter of 55 m and a rise of 12 m, still existing today. This spatial system was a high-degree statically-indeterminate structure, and it could not be easily calculated using the current structural analysis tools. Therefore, Schwedler considered to merge this spatial framework in order to form a two-dimensional curved elastic continuum to be computed: "*…in considering the dome equilibrium, it is necessary to dispense with elastic member theory, and instead use thin elastic plates in double curvature as a basis…*" [2].

Schwedler's first attempt to study two-dimensional curved structures was soon abandoned in the following years [2, 5–7], and then resumed in the first half of the 20th Century by Flügge [4, 8].

In the meanwhile, an effective development of the theory of spatial frameworks and design of shell-type spatial frameworks [4, 5] was provided by August Otto Föppl (1854-1924). From this point of view, Franz Anton Dischinger (1887-1953), chief engineer at Dywidag (Dyckerhoff & Widmann Aktien Gesellschaft) in Berlin, was a pioneer in the construction of thin domes [2, 4]. In 1922, in collaboration with Walther Bauersfeld (1879-1959), he designed a hemispherical spatial framework for the Zeiss Planetarium dome in Jena. In order to obtain a smooth projection surface for the artificial night sky, Dischinger recommended the groundbreaking use of sprayed concrete, or "Spritzbeton". This hemispherical shell, with a thickness of 30 mm and a diameter of 16 m, represents the first example of the Zeiss-

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Figure 1: Johann Wilhelm Schwedler (1823-1894)

Figure 2: Gottfried Wilhelm Flügge (1904-1990)

Dywidag System for thin domes: a spatial framework acting structurally as a reinforcement layer coated by a concrete matrix [2, 6, 7].

In 1934, Gottfried Wilhelm Flügge (1904-1990) a German engineer and professor of Mechanics at Stanford (Figure 2), published a founding treatise on shell structures [8], which can be considered as the point of departure for the following analytical and experimental research investigations [9–11]. In his book, Flügge explores the linear theory of cylindrical shells, shells of revolution and membranes, also giving extended consideration to the theory of elastic stability and vibrations.

It is interesting to note that Flügge, like Dischinger, was employed at Dywidag from 1927 to 1930, being involved in the development of reinforced concrete thin dome constructions.

After the fundamental work of Flügge, which set the stage for modern shell analysis, the theory of structures, with particular reference to shells and membranes, was rewritten and published in a more organic form by two Russian scientists [4]: Stepan Prokof'evič Timoshenko (1878-1972), and Viktor Valentinovich Novozhilov (1892-1970). The works of Timoshenko [9] and Novozhilov [10, 11] can be considered as a systematic and encyclopedic digest, gathering all the previous scientific results on shell theory [4].

Nevertheless, within the abovementioned structural theories, the concepts of funicularity and static-kinematic duality were totally absent. It is interesting to remark that the prodromes of the notion of funicularity can be acknowledged in the fundamental work by Robert Hooke [12], who was the first to realize that the ideal shape for a curved structure is that of a funicular polygon, reporting this fact in the form of an anagram with the following solution: *Ut pendet continuum flexile sic stabit contiguum rigidum inversum* (as the continuous flexible hangs downward, so will the continuous rigid stand upward inverted). We can associate to this concept the behaviour of a rope finding a shape to carry a certain load [13, 14]. For shell structures and thin domes, funicularity is representative of the internal thrust forces (without bending moments) that equilibrate the external loadings: these forces can be represented as the branches of a network following the funicular polygons [15, 16].

In order to thoroughly address the elastic problem of shell structures and thin domes static and kinematic matrix operator equations are formulated explicitly in the following, starting from beams, arches, and ropes. In this way, any possible omission present in the previous approaches becomes evident.

2 The elastic problem of beams and ropes

In the present section, the elastic problem is formulated in the details for the case of beams – with rectilinear or curvilinear axis – and of ropes, also emphasizing how the static and kinematic matrix operators are one the adjoint of the other.

As a matter of fact, static-kinematic duality leads to a simple and direct demonstration of the Principle of Virtual Work for deformable bodies, and vice-versa [17–20], the two concepts implying each other. Such a demonstration derives from the representation of the elastic problem in a symmetrical manner by combining the three fundamental relations – indefinite equations of equilibrium, kinematic equations as definitions of deformation characteristics, and constitutive equations – in a single matrix operator equation where the unknown is represented by the displacement vector.

In Figure 3, an elementary portion of a beam with rectilinear axis and a cross section that is symmetrical with respect to the vertical axis is considered. Let this portion be subjected to the shearing force, T, the axial force, N, and the bending moment, M, whereas among the components of the vector of external generalized forces appears, in addition to the distributed transverse load q and the distributed axial load, p, also the distributed bending moment, m.

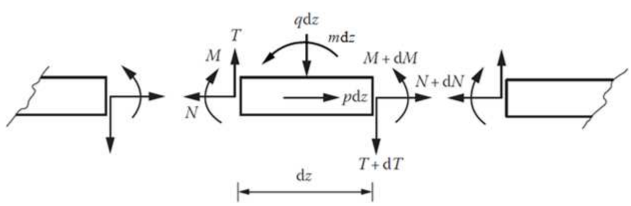

Figure 3: Beam with rectilinear axis

Thus, the indefinite equations of equilibrium result to be:

$$\frac{dT}{dz} + q = 0, \tag{1a}$$

$$\frac{dN}{dz} + p = 0, \tag{1b}$$

$$\frac{dM}{dz} - T + m = 0, \tag{1c}$$

and in operator matrix form [19]:

$$\begin{bmatrix} \frac{d}{dz} & 0 & 0 \\ 0 & \frac{d}{dz} & 0 \\ -1 & 0 & \frac{d}{dz} \end{bmatrix} \begin{bmatrix} T \\ N \\ M \end{bmatrix} + \begin{bmatrix} q \\ p \\ m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2}$$

The first equation represents the equilibrium to transverse translation, the second represents the equilibrium to axial translation, and the third the equilibrium to rotation. The matrix operator presents the total derivative d/dz in all diagonal positions, whereas the off-diagonal terms are all zero, except for one, which is equal to −1 and expresses the identity of the shear with the derivative of the bending moment (neglecting the distributed moment m).

From the matrix form, it clearly results that the elastic problem of a beam with rectilinear axis results to be internally statically-determinate, consisting of three equations in the three static unknowns T, N, M.

On the other hand, in Figure 4, the deformations in a beam with rectilinear axis due to shearing force and rigid rotation will produce relative displacements between the centroids of the two extreme cross sections of the beam portion, which are exclusively in the transverse direction of the Y axis.

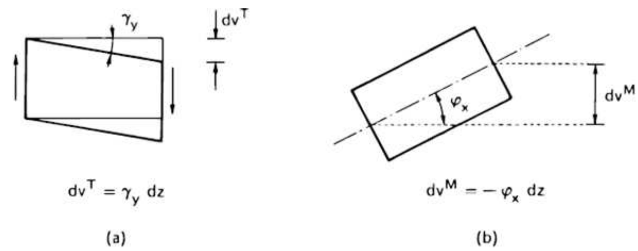

Figure 4: Beam with rectilinear axis: Relative transverse displacements in a rectilinear beam due to shearing strain (a) and rigid rotation (b)

Summing up the two significant contributions of the shearing strain and of the rigid rotation, we obtain

$$dv = dv^T + dv^M = \gamma_y dz - \varphi_x dz, \tag{3}$$

from which

$$\frac{dv}{dz} = \gamma_y - \varphi_x. \tag{4}$$

Therefore, it is possible to formulate the kinematic equations defining the characteristics of deformation as functions of the generalized displacements [19]:

$$\begin{bmatrix} \gamma_y \\ \varepsilon_z \\ \chi_x \end{bmatrix} = \begin{bmatrix} \frac{d}{dz} & 0 & +1 \\ 0 & \frac{d}{dz} & 0 \\ 0 & 0 & \frac{d}{dz} \end{bmatrix} \begin{bmatrix} v \\ w \\ \varphi_x \end{bmatrix}, \tag{5}$$

where, among the components of the deformation vector, appear the shearing strain, γ_y, the axial dilation, ε_z, and the flexural curvature, χ_x, whereas among the components of the displacement vector appears, in addition to the ordinary components, v and w, also the rotation φ_x. The transformation matrix is differential and shows on the diagonal the total derivative d/dz, whereas the off-diagonal terms are all zero except for one, which is equal to +1.

Here, the static-kinematic duality in the problem of a beam with rectilinear axis clearly emerges, being the static matrix operator equal to the transpose of the kinematic matrix operator, but for the algebraical terms, which present opposite signs. This is said to be the adjoint operator of the previous one, and vice-versa [19].

Moreover, if we consider a beam with curved axis (Figure 5), the curvilinear coordinate, s, is set as increasing as we proceed from left to right along the beam, whereas the angle $\mathrm{d}\vartheta$ is considered to be positive if it is counterclockwise.

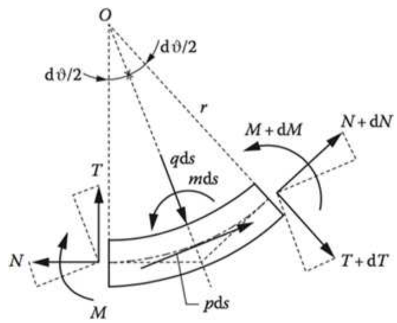

Figure 5: Beam with curvilinear axis

In accordance with the above mentioned conventions, also the radius of curvature, r, acquires an algebraic sign on the basis of the relation

$$\mathrm{d}s = r\mathrm{d}\vartheta. \tag{6}$$

Thus, the indefinite equations of equilibrium, or static equations, result to be:

$$\frac{\mathrm{d}T}{\mathrm{d}s} - \frac{N}{r} + q = 0, \tag{7a}$$

$$\frac{\mathrm{d}N}{\mathrm{d}s} + \frac{T}{r} + p = 0, \tag{7b}$$

$$\frac{\mathrm{d}M}{\mathrm{d}s} - T + m = 0, \tag{7c}$$

and in matrix form [19]:

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}s} & -\frac{1}{r} & 0 \\ \frac{1}{r} & \frac{\mathrm{d}}{\mathrm{d}s} & 0 \\ -1 & 0 & \frac{\mathrm{d}}{\mathrm{d}s} \end{bmatrix} \begin{bmatrix} T \\ N \\ M \end{bmatrix} + \begin{bmatrix} q \\ p \\ m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0S \end{bmatrix}. \tag{8}$$

Also in this case, from the operator matrix form, it clearly results that the elastic problem of a beam with curved axis results to be internally statically-determinate, and it is described by three equations in the three static unknowns T, N, M.

As regards the kinematic problem of a beam with curved axis (Figure 6), the tangential displacement, w, produces an apparent rotation $\varphi(w)$, which is to be subtracted from the total rotation to obtain the true rotation (Figure 6a):

$$\varphi(w) = \frac{w}{r}. \tag{9}$$

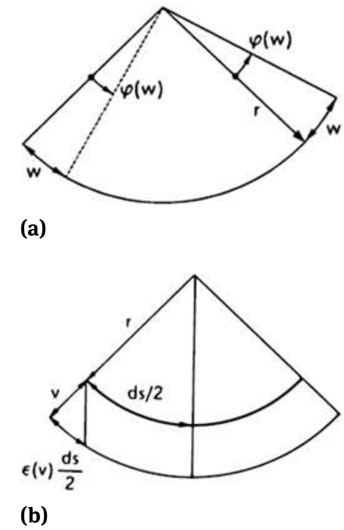

Figure 6: Beam with curvilinear axis: rigid rotation to be subtracted from the total one and due to a uniform tangential displacement (a); additional axial dilation due to a uniform radial displacement (b)

On the other hand, the radial displacement v produces an additional axial dilation $\varepsilon(v)$ which is given by (Figure 6b):

$$\varepsilon(v) = \frac{v}{r}. \tag{10}$$

As a consequence of an infinitesimal relative rotation $\mathrm{d}\varphi$ of the extreme cross sections of the beam element, the angle between the sections can be obtained as the sum $(\mathrm{d}\vartheta + \mathrm{d}\varphi)$ of the initial and intrinsic relative rotation with the elastic and flexural relative rotation. The new curvature is then

$$\chi_{total} = \frac{\mathrm{d}\vartheta + \mathrm{d}\varphi}{\mathrm{d}s}, \tag{11}$$

so that the variation of curvature is

$$\chi = \chi_{total} - \frac{1}{r} = \frac{\mathrm{d}\varphi}{\mathrm{d}s}. \tag{12}$$

Then, considering the rotation to be deducted, the additional axial dilation, and the variation in curvature, the kinematic equations for the curved beam appear as follows:

$$\begin{bmatrix} \gamma \\ \varepsilon \\ \chi \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}s} & -\frac{1}{r} & +1 \\ \frac{1}{r} & \frac{\mathrm{d}}{\mathrm{d}s} & 0 \\ 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}s} \end{bmatrix} \begin{bmatrix} v \\ w \\ \varphi \end{bmatrix}. \tag{13}$$

It should be noted that, but for the algebraic signs of the non-differential terms, the static matrix is the transpose of the kinematic one, and vice versa.

Moreover, when the flexural stiffness of the beam vanishes, *i.e.*, $M = T = 0$ by hypothesis, the elastic problem of a rope arises (Figure 7), and the static equations become:

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}s} & -\frac{1}{r} & 0 \\ \frac{1}{r} & \frac{\mathrm{d}}{\mathrm{d}s} & 0 \\ -1 & 0 & \frac{\mathrm{d}}{\mathrm{d}s} \end{bmatrix} \begin{bmatrix} 0 \\ N \\ 0 \end{bmatrix} + \begin{bmatrix} q \\ p \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{14}$$

obtaining two equations in the single unknown N (hypostatic system),

$$\frac{N}{r} = q, \qquad (15a)$$

$$\frac{\mathrm{d}N}{\mathrm{d}s} = -p, \qquad (15b)$$

and entailing the coincidence of the configuration of the structural element with its thrust line (Figure 7).

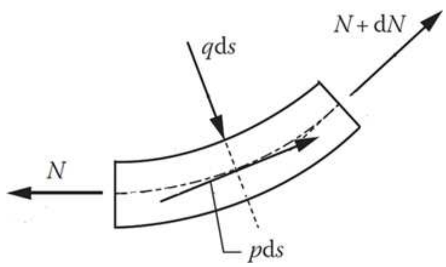

Figure 7: Rope element in equilibrium

Hence, it can be affirmed that, moving from beam to rope, *i.e.*, reducing the flexural stiffness, the elastic problem shifts from internally isostatic to hypostatic.

3 The elastic problem of shells and membranes

When a shell of revolution is loaded symmetrically with respect to the axis of symmetry Z (Figure 8), the static equations may be written as [19]:

$$\begin{bmatrix} \left(\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\sin\alpha}{r}\right) & -\frac{\sin\alpha}{r} & \frac{1}{R_1} & 0 & 0 \\ -\frac{1}{R_1} & -\frac{1}{R_2} & \left(\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\sin\alpha}{r}\right) & 0 & 0 \\ 0 & 0 & -1 & \left(\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\sin\alpha}{r}\right) & -\frac{\sin\alpha}{r} \end{bmatrix} \begin{bmatrix} N_s \\ N_\vartheta \\ T_s \\ M_s \\ M_\vartheta \end{bmatrix} + \begin{bmatrix} p_s \\ q \\ m_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (16)$$

where N_s and N_ϑ are the membrane forces along the meridian and the parallel, respectively, T_s is the shearing force along the meridian, M_s and M_ϑ are the bending moments about the parallel and the meridian, respectively. The first equation represents the equilibrium to translation along the meridian, the second represents the equilibrium to translation along the normal, and the third the equilibrium to rotation about the parallel. Note that, for reasons of symmetry, the conditions of equilibrium to translation along the parallel and to rotation about the meridian are identically satisfied and thus do not appear in the previous equations.

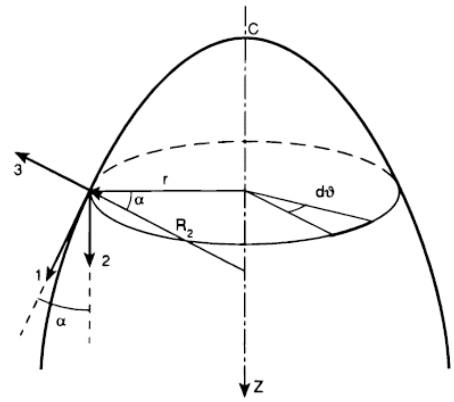

Figure 8: Shell of revolution or thin dome

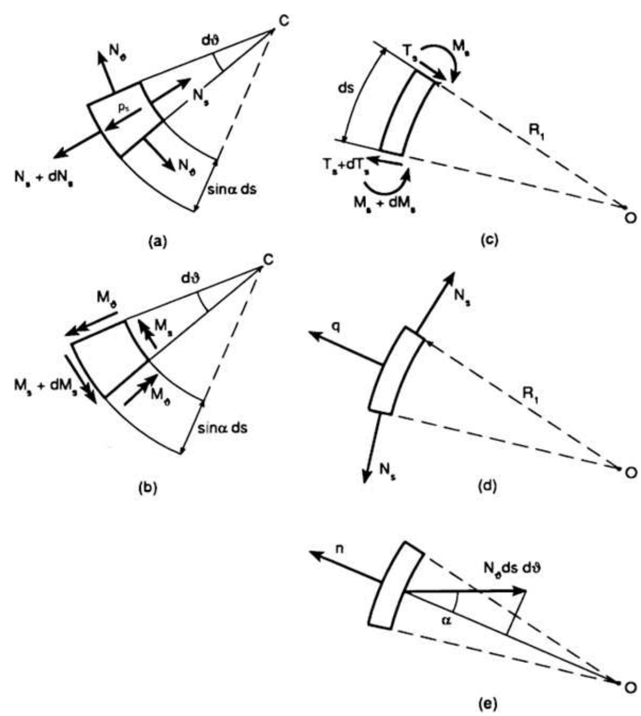

Figure 9: Equilibrium of a shell of revolution: (a) translation along the meridian; (b,c) rotation about the parallel; (d,e) translation along the normal to the surface

It can be deduced that the elastic problem of a shell of revolution presents two degrees of internal redundancy, consisting of three equations of equilibrium in the five static unknowns N_s, N_ϑ, T_s, M_s, M_ϑ, whereas the more general problem of shells with double curvature appears to have three degrees of internal redundancy.

As regards the kinematic equations of a shell of revolution, applying the virtual work principle we can obtain [18, 20]:

$$\begin{bmatrix} \varepsilon_s \\ \varepsilon_\vartheta \\ \gamma_s \\ \chi_s \\ \chi_\vartheta \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}s} & \frac{1}{R_1} & 0 \\ +\frac{\sin\alpha}{r} & \frac{1}{R_2} & 0 \\ -\frac{1}{R_1} & \frac{\mathrm{d}}{\mathrm{d}s} & +1 \\ 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}s} \\ 0 & 0 & +\frac{\sin\alpha}{r} \end{bmatrix} \begin{bmatrix} u \\ w \\ \varphi_s \end{bmatrix}, \tag{17}$$

where ε_s and ε_ϑ are the membrane dilations, γ_s is the shearing strain along the meridian, χ_s and χ_ϑ are the curvatures, as well as u is the displacement along the meridian, w the normal displacement, and φ_s the rotation about the parallel.

It is interesting to recall that Timoshenko in 1940 [9], considering the kinematic equations of a shell of revolution symmetrically loaded with respect to the axis of symmetry Z, neglected the shearing strain, not obeying in this way the static-kinematic duality:

$$\begin{bmatrix} \varepsilon_s \\ \varepsilon_\vartheta \\ \gamma_s \\ \chi_s \\ \chi_\vartheta \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}s} & \frac{1}{R_1} & 0 \\ +\frac{\sin\alpha}{r} & \frac{1}{R_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}s} \\ 0 & 0 & +\frac{\sin\alpha}{r} \end{bmatrix} \begin{bmatrix} u \\ w \\ \varphi_s \end{bmatrix}. \tag{18}$$

The differential matrix operator is obtained here by reproposing in a matrix form the kinematic equations as they appear in [9]. Furthermore, in 1947 also Novozhilov [10, 11] proposed the same kinematic formulation for the shell of revolution as that by Timoshenko [4]. Note that both Timoshenko and Novozhilov, following Flügge [8], disregarded the deformation γ_s in the kinematic problem of a shell of revolution [4]. Thus, it is important to consider that the terms of the kinematic matrix operator are correctly determined only referring to the complete kinematic theory provided by the static-kinematic duality and, therefore, by the virtual work principle [4, 19]. On the contrary, considering the incomplete kinematic theory based on the foregoing classical approaches by Flügge [8], Timoshenko [9], and Novozhilov [10, 11], some inaccuracy may arise in the analysis of the shells of revolution when dealing with the shearing deformation. It must be also pointed out that a comparison between complete and incomplete kinematic formulations has been proposed recently by Elishakoff [21, 22], concerning the Uflyand-Mindlin theory of elastic plates [23, 24] as an extension of the Kirchhoff-Love theory [25, 26], taking into account shear deformations through the thickness of the plate. On the other hand, from a static point of view, Reissner theory [27] can be considered as the complete static formulation of the plate theory, which was introduced earlier by Kirchhoff and Love in incomplete terms [18, 20–22, 25–29].

Eventually, we consider shells of such a small thickness to present an altogether negligible flexural stiffness. These elements can sustain only compressive or tensile forces contained in their tangent planes. The kinematic and static equations of thin domes simplify notably, since only forces along the meridians and the parallels, N_s and N_ϑ, are present, as well as the displacements along the meridians and those normal to the middle surface, u and w, respectively (Figure 8):

$$\begin{bmatrix} \left(\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\sin\alpha}{r}\right) & -\frac{\sin\alpha}{r} \\ -\frac{1}{R_1} & -\frac{1}{R_2} \end{bmatrix} \begin{bmatrix} N_s \\ N_\vartheta \end{bmatrix} + \begin{bmatrix} p_s \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{19}$$

$$\begin{bmatrix} \varepsilon_s \\ \varepsilon_\vartheta \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}s} & \frac{1}{R_1} \\ \frac{\sin\alpha}{r} & \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}. \tag{20}$$

The first static equation represents the equilibrium to translation along the meridian, whereas the second represents the equilibrium to translation along the normal [19, 20].

It can be concluded that the elastic problem of a thin dome is internally statically-determinate, consisting of two equations in the two static unknowns N_s and N_ϑ.

Hence, in analogy to the case of beams and ropes, moving from shell to membrane, *i.e.*, tending the flexural stiffness to zero, the problem shifts from internally twice hyperstatic to isostatic. In other terms, reducing the shell thickness, a funicular regime prevails, the constraint reactions being assumed to be tangential to meridians at the dome edges: independently of the shallowness ratio or the shape, the thin dome sustains only compressive or tensile forces contained in its tangent plane. On the contrary, when the thickness is finite, the static problem of the dome turns out to be governed by two independent parameters: the dome thickness and its shallowness ratio. For each dome thickness, an optimal shape exists, which minimizes the flexural regime if compared to the membrane one. Moreover, if a dome structure with non-negligible thickness is properly supported at its boundaries, it can support any load primarily by membrane action. On the other hand, bending or twisting moments and shear forces acting perpendicular to the dome will occur locally in the region of boundaries and of concentrated loads [14, 30, 31].

4 Conclusions

By means of the static-kinematic duality, the problem of one- and two-dimensional structural elements can be addressed in a comprehensive way. In particular, a beam with rectilinear or curved axis results to be internally isostatic, being characterized by a system of three equilibrium equations in three static unknowns, whereas, reducing its flexural stiffness, the hypostatic problem of a rope appears, characterized by two equilibrium equations in one single unknown. On the other hand, the problem represented by a shell of revolution is proved to be internally hyperstatic, as it is described by three equations in five static unknowns. In close analogy to the case of beams and ropes, tending the flexural stiffness to zero, a shell turns into a membrane, which results to be internally statically-determinate, being characterized by a system of two equilibrium equations in two static unknowns. Thus, differently from the theory of ropes, the solution can be obtained just based on the equilibrium equations and independently of the shape of the membrane itself. Then, the structural behavior of an elastic dome results to be governed by its shallowness ratio and its thickness, the constraint reactions being assumed to be tangential to meridians at the dome edges. On the other hand, when the dome thickness tends to zero, the funicularity emerges and prevails, independently of the shallowness ratio or the shape of the dome. When the thickness is finite, an optimal shape exists, minimizing the flexural regime if compared to the membrane one: in this case, bending or twisting moments and shear forces acting perpendicular to the dome will occur locally in the region of boundaries and of concentrated loads.

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