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Dimensional Analysis of Critical Phenomena: Self-Weight Failure, Turbulence, Resonance, Fracture

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Abstract—Both in physics and engineering, Buckingham's Π theorem is considered as a key tool for dimensional analysis, providing a method to identify the dimensionless parameters governing physical similitude and scale modeling, when the form of the solving equations is still unknown. In the present paper, the scaling of different critical phenomena in solid and fluid mechanics is emphasized by the application of the Π theorem. In particular, self-weight failure, turbulence, resonance, and fracture are considered, highlighting how these critical phenomena are governed by specific dimensionless numbers, which are functions of few fundamental mechanical quantities including the size-scale, which are characterized by different and algebraically independent physical dimensions.

Keywords: Buckingham's Π theorem, self-weight failure, turbulence, resonance, fracture

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1. INTRODUCTION

Within the scientific advancements in the field of materials mechanics, it has been realized that the classical concept of strength, understood as force per unit surface causing failure, is in need of revision, especially in the cases where particularly large or particularly small structures are involved. The strength of the material must be compared against other characteristics, such as the toughness in the case of fracturing processes, in order to define, via the structural size, the ductility or the brittleness of the structure [1–6]. The scaling as the competition between two different collapses governed by generalized forces with different physical dimensions should not be confused with the simple and very well-known thickness effect, which considers the fracture toughness variations with the specimen thickness, keeping the remaining planar sizes constant. The latter derives from a transition between plane stress and plane strain conditions. In metals, the two effects are both present and then interacting, whereas in concrete-like materials only the scaling effect is present, without any possibility of mistaking. This is the reason why scale effects were firstly studied and understood in the scientific community of concrete materials and structures.

Two intrinsic characteristics of the material, plus a geometrical characteristic of the structure, represent the minimum basis for being able to predict the type of structural response [7]. Since in structures prevalently subjected to compressive forces, a transition can be detected from plastic collapse to buckling instability as slenderness increases, so in structures subjected to tensile forces, there is a transition from plastic collapse to brittle fracture as the size-scale increases [1, 3].

Two well-known extreme cases of the abovementioned properties are represented by Liberty ships and glass filaments [8]. During Second World War, Liberty ships suddenly split into two parts, showing extremely brittle fractures without any slightest forewarning evidence. At that time, profound astonishment in technicians and researchers was due to the contrast between the extreme brittleness of the failures and the considerable ductility shown in the laboratory by specimens of the same steel [9, 10]. On the other hand, considering a microscopic filament of glass used for fibre reinforcement, it has been proven to bear large strains and stresses as much as two orders of magnitude greater than the tensile strength of the glass itself, when measured at the laboratory scale with specimens of normal dimensions. These two examples starkly remark how both strength and ductility are functions of the structural scale: brittleness and low strength can characterize enormous steel structures, as well as ductility and high strength arises in microscopic structures made of glass. On the contrary, it is well-known that, at the laboratory scale, steel is a particularly ductile material and glass a particularly brittle one [7].

Establishing scaling phenomena was always considered as crucial both in physics and engineering [11–17], in particular when qualitative solutions illuminating complex problems are needed [18–21].

In the framework of solid and fluid mechanics, the most recurrent explanations to scale dependent physical phenomena, such as structural brittleness, are based on dimensional analysis [1, 3, 6, 22], and particularly on the Π theorem by Edgar Buckingham.

2. BUCKINGHAM'S THEOREM OF DIMENSIONAL ANALYSIS (1914)

Both in physics and engineering, Buckingham's Π theorem [23] is considered as a key tool for dimensional analysis, providing a method to identify the dimensionless parameters governing the problem, even if the form of the solving equations is still unknown.

An early attempt for the formulation of the Π theorem was firstly conceived by the French mathematicians J. Bertrand [24] and A. Vaschy [25], who considered particular problems in thermodynamics and electromagnetism only. Bertrand, in his work, determined a priori the fundamental independent variables: "Length, time, and force units of measure are arbitrary and independent. When they are considered, all the other variables rely on them and can be deduced by them". Then, more explicit formulations of the dimensional analysis theorem were given by Vaschy [25], and by the Russian scientists A. Federman [26] and D.P. Riabouchinsky [27]. Finally, it was only Buckingham's work [24] that introduced the generalization of the Π theorem and the use of the symbol Π for the dimensionless parameters, from which the theorem denomination [28–32].

Let us consider the following fundamental mechanical quantities [3]: length $[L]$, force $[F]$, and time $[T]$. Consider then a certain quantity $[Q]=[L]^{\alpha}[F]^{\beta}\times[T]^{\gamma}$. If the length unit of measure is multiplied by λ, the force unit of measure by φ, and the time unit of measure by τ, it follows that Q is multiplied by $\lambda^{\alpha}\varphi^{\beta}\tau^{\gamma}$. Then, we can define three mechanical quantities Q_1, Q_2, Q_3 as follows:

$$\begin{aligned}[Q_1]&=[L]^{\alpha_1}[F]^{\beta_1}[T]^{\gamma_1},\\ [Q_2]&=[L]^{\alpha_2}[F]^{\beta_2}[T]^{\gamma_2},\\ [Q_3]&=[L]^{\alpha_3}[F]^{\beta_3}[T]^{\gamma_3}.\end{aligned} \tag{1}$$

If the units of measure (L, F, T) are multiplied by λ, φ, τ, then the units measuring Q_1, Q_2, Q_3 are multiplied by χ_1, χ_2, χ_3 so that

$$\chi_1=\lambda^{\alpha_1}\varphi^{\beta_1}\tau^{\gamma_1},\ \chi_2=\lambda^{\alpha_2}\varphi^{\beta_2}\tau^{\gamma_2},\ \chi_3=\lambda^{\alpha_3}\varphi^{\beta_3}\tau^{\gamma_3}. \tag{2}$$

It follows that

$$\begin{aligned}\ln\chi_1&=\alpha_1\ln\lambda+\beta_1\ln\varphi+\gamma_1\ln\tau,\\ \ln\chi_2&=\alpha_2\ln\lambda+\beta_2\ln\varphi+\gamma_2\ln\tau,\\ \ln\chi_3&=\alpha_3\ln\lambda+\beta_3\ln\varphi+\gamma_3\ln\tau,\end{aligned} \tag{3}$$

which is a linear system of equations with unknowns in $\ln\lambda$, $\ln\varphi$, and $\ln\tau$. It admits one and only one solution, if and only if the coefficient matrix determinant is different from zero

$$D=\begin{vmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{vmatrix}\neq 0. \tag{4}$$

When satisfied, Eq. (4) makes the three quantities Q_1, Q_2, and Q_3 dimensionally independent. They may thus be regarded as fundamental quantities.

In addition, another equivalent definition of dimensionally independent quantities can be invoked [3]. The three quantities Q_1, Q_2, and Q_3 are dimensionally independent when any quantity $[Q_0]=[L]^{\alpha_0}\times[F]^{\beta_0}[T]^{\gamma_0}$ can have the same physical dimensions as the product $Q_1^{\alpha_{10}}Q_2^{\alpha_{20}}Q_3^{\alpha_{30}}$ for appropriate values of α_{10}, α_{20} and α_{30}. From Eqs. (1) it follows that

$$\begin{aligned}\alpha_0&=\alpha_1\alpha_{10}+\alpha_2\alpha_{20}+\alpha_3\alpha_{30},\\ \beta_0&=\beta_1\alpha_{10}+\beta_2\alpha_{20}+\beta_3\alpha_{30},\\ \gamma_0&=\gamma_1\alpha_{10}+\gamma_2\alpha_{20}+\gamma_3\alpha_{30},\end{aligned} \tag{5}$$

which is a linear system with unknowns α_{10}, α_{20}, α_{30}, and the transpose matrix D^{T} of the system in Eqs. (3) as coefficient matrix. Equations (3) and (5) admit of one and only one solution, if and only if the condition in Eq. (4) holds.

There are only two fundamental quantities in the static field: length $[L]$ and force $[F]$, since time is not involved. According to these considerations, it is possible to state that stress $[\sigma]=[L]^{-2}[F]$, length $[L]$, and time $[T]$ are dimensionally independent:

$$D=\begin{vmatrix}-2 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{vmatrix}=-1\neq 0. \tag{6}$$

Thus, the identified fundamental quantities can be suitably combined to determine the Π sets of dimensionless parameters, as in the following.

Let q_0 be the critical load at final failure for a structural element. This load is a function of several variables and can be expressed as

$$q_0 = F(q_1, q_2, ..., q_n; r_1, r_2, ..., r_m), \quad (7)$$

where q_i are the physical quantities with different dimensions and r_i are nondimensional numbers as the geometrical ratios. Each quantity with certain dimensions appears just once in the function F. For example, q_1 could be the ultimate strength σ_u, q_2 could be the reference size L of the structure; q_3 could be, for example, the fracture toughness K_{IC}, or the Young's modulus E, r_1, r_2, r_3, etc., the ratios of the other body sizes, which are characteristic of the structural geometry, to the size L. Consider two dimensionally independent quantities q_1 and q_2. They are considered as fundamental quantities such that the product $q_1^{\alpha_{10}} q_2^{\alpha_{20}}$ has the same dimensions as q_0 for suitable values of α_{10} and α_{20}. In the same way, the product $q_1^{\alpha_{13}} q_2^{\alpha_{23}}$ can have the same dimensions as q_3, for suitable values of α_{13} and α_{23}, and so on. The function in Eq. (7) can therefore be transformed into

$$\frac{q_0}{q_1^{\alpha_{10}} q_2^{\alpha_{20}}} = \Pi\left(\frac{q_3}{q_1^{\alpha_{13}} q_2^{\alpha_{23}}}, ..., \frac{q_n}{q_1^{\alpha_{1n}} q_2^{\alpha_{2n}}}; r_1, r_2, ... r_m\right). \quad (8)$$

Function F becomes Π because of the nondimensionalization [23]. If the unit of measure of q_1 changes Π, being a dimensionless number, does not vary. Therefore, Π is not really a function of q_1 nor of q_2. It is only a function of $(n-2+m)$ dimensionless numbers, and thus

$$\Pi_0 = \Pi(\Pi_3, ..., \Pi_n; r_1, r_2, ..., r_m). \quad (9)$$

3. SCALING OF CRITICAL PHENOMENA IN SOLID AND FLUID MECHANICS

3.1. Scaling of Self-Weight Failure

The concept of scaling in solid mechanics is not new. In his fundamental work [33], Galileo Galilei discussed the size-scale effect in relation to material strength of bodies subjected to their self-weight. He observed that small models are proportionally much stronger than full-scale bodies. If only the shape were responsible for the strength, then any object could be reproduced at any scale level without sacrificing its strength. Quoting from his work: "a horse will break its bone when falling from a height of three arms, whereas a cat will not be injured falling from a height of eight or ten, neither a cricket from a tower, nor an ant falling from the moon". Galileo continues to argue that: "Nature could not make trees of enormous magnitude because the branches would then break due to their self-weight ... nor giant men and animals, unless much harder and stronger materials or much less slender bones were used" [33].

Galileo considered the failure of solid bodies subjected to their self-weight P by applying a stress criterion. If L is a characteristic size of the body and it is allowed to vary, whereas the geometrical shape is kept constant, and A is the body cross section, then $P \sim [L]^3$, $A \sim [L]^2$, and $\sigma = P/A \sim [L]$. Since P increases with the volume, the stress due to self-weight increases proportionally to L, and hence the loading capacity of the body can be scaled in an inversely proportional manner with respect to L [34].

In this regard, Buckingham's theorem can be applied to analyze the above described simple system, consisting of a body loaded by its self-weight. The maximum stress σ_{max} acting within the body can be expressed as

$$\sigma_{max} = F((\rho g), L), \quad (10)$$

where ρ is the material density, $[\rho] = [M][L]^{-3}$, g is the gravity acceleration, $[g] = [L][T]^{-2}$, and M indicates the body mass. Knowing that $[M][L][T]^{-2} = [F]$, then $[\rho g] = [F][L]^{-3}$.

If (ρg) and L are regarded as fundamental quantities, then Eq. (10) gives the scaling of self-weight failure:

$$\sigma_{max} = (\rho g) L \Pi, \quad (11)$$

where Π is a dimensionless constant depending only on structural shape and external constraints. Equation (11) describes the abovementioned Galilean scale-effect: the maximum stress due to body self-weight can be scaled in a directly proportional manner with regard to the characteristic size L.

3.2. Turbulence Scaling

The presence of a competition between laminar and turbulent flow in fluid mechanics suggests extending to this field the dimensional analysis considerations [11, 35, 36]. The occurrence of turbulence can be analyzed using Buckingham's mathematical formalism and assuming that the critical velocity v_0 for the appearance of turbulence in fluid motion can be expressed as

$$v_0 = F((\rho g), \nu, L), \quad (12)$$

where ν is the kinematic viscosity, $[\nu] = [L]^2[T]^{-1}$.

If (ρg) and L are regarded as fundamental quantities, then Eq. (12) returns the turbulence scaling:

$$v_0 = (gL)^{1/2}\Pi\left(\frac{\nu}{g^{1/2}L^{3/2}}\right). \quad (13)$$

Function Π, which is obtained through Buckingham's theorem, turns out to govern the transition from laminar to turbulent flow. On the other hand, Reynolds [35] implicitly assumed the specific function $\Pi(x) = \Pi x$, where Π is a dimensionless constant. The reason for this assumption is that

$$\lim_{\nu\to 0} v_0 = 0.$$

Based on this choice, we have

$$\frac{v_0 L}{\nu} = \Pi = \mathrm{Re} \sim 2200. \quad (14)$$

3.3. Resonance Scaling

As a third case of critical phenomenon, let us consider an elastic body subjected to an oscillating loading. In structural engineering, the phenomenon of resonance represents a particular form of global collapse, which occurs when an external periodic frequency matches one of the natural frequencies of vibration of the mechanical system [37].

Therefore, the critical value of external periodic frequency ω_0, leading to resonance, is represented by the natural frequency of the body itself, which can be expressed as

$$\omega_0 = F((\rho g), L, E), \quad (15)$$

where E is the Young's modulus of the material, $[E] = [F][L]^{-2}$.

If (ρg) and L are regarded as fundamental quantities, then Eq. (15) gives the resonance scaling:

$$\omega_0 = \left(\frac{g}{L}\right)^{1/2}\Pi\left(\frac{E}{(\rho g)L}\right). \quad (16)$$

Observing that

$$\lim_{E\to 0} \omega_0 = 0,$$

it is possible to assume the specific function $\Pi(x) = \Pi x$, where Π is a dimensionless constant depending only on structural shape and external constraints. Based on this choice, we have

$$\frac{\omega_0 \rho}{E} L^{3/2} g^{1/2} = \Pi. \quad (17)$$

It is interesting to remark that, on the left-hand side of Eq. (17), a particular dimensionless number appears where natural frequency, mass density of the material, Young's modulus of the material, power of the size-scale of the body with exponent equal to 3/2, and power of the gravity acceleration with exponent equal to 1/2 are included. In addition to the natural frequency, a generalized mass ρ and a generalized stiffness E emerge, as usually occurs in dynamics problems.

3.4. Fracture Scaling

The mechanical behavior of a structural element and its brittle fracture can be reproduced by a model only if its size-scale is not too small [3]. In the case of an elastic-plastic cracked body, two different failure modes are possible: (i) plastic flow collapse of the structure (σ_P—yield strength of the material), with the crack considered as a weakening of the cross section without including any local effect; (ii) crack propagation determined by the achievement of the fracture toughness of the material K_{IC}. Thus, the two generalized forces involved in the abovementioned failures are $[\sigma] = [L]^{-2}[F]$ and $[K_I] = [L]^{-3/2}[F]$, the dimension mismatch between them being evident.

The critical load q_0 is given by

$$q_0 = F(\sigma_P, L, K_{IC}; a/L), \quad (18)$$

where the crack length a is also included. If σ_P and L are regarded as fundamental quantities, then Eq. (18) leads to the fracture scaling:

$$q_0 = \sigma_P L^{\alpha}\Pi\left(\frac{K_{IC}}{\sigma_P L^{1/2}}; \frac{a}{L}\right), \quad (19)$$

where $\alpha = -1, 0, 1, 2, 3$. Let us observe that the critical load dimensions can range from those of a body force $[q_0] = [F][L]^{-3}$, when $\alpha = -1$, to those of an applied moment $[q_0] = [F][L]$, when $\alpha = 3$.

Value Π governs the ductile-to-brittle transition and it is a function of the brittleness number

$$s = \frac{K_{IC}}{\sigma_P L^{1/2}} \quad (20)$$

and of the relative crack depth a/L [1, 3]. Both mechanical properties of the material and size of the structure appear in the number s. It is possible to demonstrate that brittle failure occurs only with relatively low fracture toughnesses K_{IC}, high yield strengths σ_P, and/or large structural sizes L, i.e. when s is lower than its threshold value s_0 [3].

REFERENCES

1. Carpinteri, A., Size Effect in Fracture Toughness Testing: A Dimensional Analysis Approach, in *Proceedings of the International Conference on Analytical and Experimental Fracture Mechanics, Roma, Italy, 1980*, pp. 785–797.
2. Carpinteri, A., A Fracture Mechanics Model for Reinforced Concrete Collapse, in *Proceedings of the IABSE Colloquium on Advanced Mechanics of Reinforced Concrete*, Delft, 1981, pp. 17–30.

3. Carpinteri, A., Notch Sensitivity in Fracture Testing of Aggregative Materials, *Eng. Fract. Mech*., 1982, vol. 16, pp. 467–481.
4. Carpinteri, A., Stability of Fracturing Process in RC Beams, *J. Struct. Eng. ASCE*, 1984, vol. 110, pp. 544–558.
5. Carpinteri, A., Interpretation of the Griffith Instability as a Bifurcation of the Global Equilibrium, in *Application of Fracture Mechanics to Cementitious Composites. Proceedings of a NATO Advanced Research Workshop, Evanston, USA, 1984*, Shah, S.P., Ed., Dordrecht: Martinus Nijhoff Publishers, 1985, pp. 287–316.
6. Carpinteri, A., Cusp Catastrophe Interpretation of Fracture Instability, *J. Mech. Phys. Solids*, 1989, vol. 37, pp. 567–582.
7. Carpinteri, A., *Structural Mechanics: A Unified Approach*, London: Chapman & Hall, 1997.
8. Carpinteri, A., Cornetti, P., and Puzzi, S., Scaling Laws and Multiscale Approach in the Mechanics of Heterogeneous and Disordered Materials, *ASME Appl. Mech. Rev*., 2006, vol. 59, pp. 283–305.
9. Tipper, C., Brittle Fracture of Mild Steel Plates, in *British Iron and Steel Research Association Proceedings*, Cambridge, 1945, pp. 23–50.
10. Tipper, C., *The Brittle Fracture Story*, Cambridge: Cambridge University Press, 1962.
11. Barenblatt, G.I., *Flow, Deformation and Fracture*, Cambridge: Cambridge University Press, 2014.
12. Barenblatt, G.I. and Botvina, L.R., A Note Concerning Power-Type Constitutive Equations of Deformation and Fracture of Solids, *Int. J. Eng. Sci*., 1982, vol. 20, pp. 187–191.
13. Carpinteri, A., Scaling Laws and Renormalization Groups for Strength and Toughness of Disordered Materials, *Int. J. Solids Struct*., 1994, vol. 31, pp. 291–302.
14. Carpinteri, A. and Pugno, N., Are Scaling Laws on Strength of Solids Related to Mechanics or to Geometry? *Nature Mater*., 2005, vol. 4, pp. 421–423.
15. Carpinteri, A. and Accornero, F., Rotation versus Curvature Fractal Scaling in Bending Failure, *Phys. Mesomech*., 2019, vol. 22, no. 1, pp. 46–51. doi 10.1134/S1029959919010089
16. Goldstein, R.V. and Vainshelbaum, V.M., Material Scale Length as a Measure of Fracture Toughness in Fracture Mechanics of Plastic Materials, *Int. J. Fracture*, 1978, vol. 14, pp. 185–201.
17. Wilson, K.G., Renormalization Group and Critical Phenomena, *Phys. Rev. B*, 1971, vol. 4, pp. 3174–3183.
18. Panin, V.E., Overview on Mesomechanics of Plastic Deformation and Fracture of Solids, *Theor. Appl. Fract. Mech*., 1998, vol. 30(1), pp. 1–11.
19. Panin, V.E., Synergetic Principles of Physical Mesomechanics, *Theor. Appl. Fract. Mech*., 2001, vol. 37(1–3), pp. 261–298.
20. Panin, V.E. and Egorushkin, V.E., Deformable Solid as a Nonlinear Hierarchically Organized System, *Phys. Mesomech*., 2011, vol. 14, no. 5–6, pp. 207–223.
21. Panin, V.E., Korotaev, A.D., Makarov, P.V., and Kuznetsov, V.M., Physical Mesomechanics of Materials, *Russ. Phys. J*., 1998, vol. 41(9), pp. 856–884.
22. Carpinteri, A. and Paggi, M., Singular Harmonic Problems at a Wedge Vertex: Mathematical Analogies Between Elasticity, Diffusion, Electromagnetism, and Fluid Dynamics, *J. Mech. Mater. Struct*., 2011, vol. 6, pp. 1–4.
23. Buckingham, E., On Physically Similar Systems: Illustrations of the Use of Dimensional Equation, *Phys. Rev*., 1914, vol. 4(4), pp. 345–376.
24. Bertrand, J., Sur l'homogénéité dans les formules de physique, *Comptes Rendus*, 1878, vol. 86(15), pp. 916–920.
25. Vaschy, A., Sur les lois de similitude en physique, *Annales Télégraphiques*, 1892, vol. 19, pp. 25–28.
26. Federman, A., On Some General Methods of Integration of First-Order Partial Differential Equations, *Proc. Saint-Petersburg Polytechnic Institute*, 1911, vol. 16(1), pp. 97–155.
27. Riabouchinsky, D.P., Méthode des variables de dimension zéro et son application en aérodynamique, *L'Aérophile*, 1911, pp. 407–408.
28. Birkhoff, G., *Hydrodynamics: A Study in Logic, Fact, and Similitude*, Princeton University Press, 1956.
29. Borodich, F.M., Scaling Transformations in Solid Mechanics, in *IUTAM Symposium on Scaling in Solid Mechanics*, Springer, 2009, pp. 11–26.
30. Borodich, F.M., History of the Π Theorem, in *Advances in Applied Mechanics*, Elsevier, 2014, pp. 283–284.
31. Bridgman, P.W., *Dimensional Analysis*, New Haven: Yale University Press, 1931.
32. Curtis, W.D., Logan, J.D., and Parker, W.A., Dimensional Analysis and the Pi Theorem, *Linear Algebra and Its Applications*, 1982, vol. 47, pp. 117–126.
33. Galilei, G., *Discorsi e dimostrazioni matematiche intorno a due nuove scienze attenenti alla mecanica e i movimenti locali*, Leiden, The Netherlands: Ludovico Elzeviro, 1638.
34. Carpinteri, A., *Mechanical Damage and Crack Growth in Concrete: Plastic Collapse to Brittle Fracture*, Dordrecht: Martinus Nijhoff Publishers, 1986.
35. Reynolds, O., An Experimental Investigation of the Circumstances which Determine whether the Motion of Water Shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels, *Philos. Trans. Roy. Soc*., 1883, vol. 174, pp. 935–982.
36. Kolmogorov, A.N., Dissipation of Energy in Locally Isotropic Turbulence, *Proc. USSR Acad. Sci*., 1941, vol. 32, pp. 16–18.
37. Timoshenko, S.P., *Vibration Problems in Engineering*, Van Nostrand, 1928.