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Scaling and fractality in subcritical fatigue crack growth: Crack-size effects on Paris' law and fatigue threshold

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Abstract
The present contribution investigates the crack-size effects on Paris' law in accordance with dimensional analysis and intermediate asymptotics theory, which makes it possible to obtain a generalised equation able to provide an interpretation to the various empirical power-laws available in the Literature. Subsequently, within the framework of fractal geometry, scaling laws are determined for the coordinates of the limit-points of Paris' curve so that a theoretical explanation is provided to the so-called short cracks problem. Eventually, the proposed models are compared with experimental data available in the literature which seem to confirm the advantage of applying a fractal model to the fatigue problem.

KEYWORDS
crack-size effects, dimensional analysis, fatigue threshold, fractal geometry, intermediate asymptotics, Paris' law

1 | INTRODUCTION

Paris, in the 1960s, made a major breakthrough on the fatigue problem using an analytical approach based on linear elastic fracture mechanics (LEFM).[1] In fact, fatigue failure occurs because of the propagation of a pre-existing crack, which is contained in the structural element. Paris showed that the crack growth rate, da/dN, results to be a function of the stress-intensity factor range, ΔK_I. He found that a power-law relationship is able to describe the propagation of cracks in good agreement with experimental data available in the literature as follows:

$$da/dN = C\,\Delta K_I{}^m. \tag{1}$$

Actually, Paris' law is only valid in the central part of the diagram (Region II) because this power-law relationship presents some deviations (see Figure 1). In fact, when the stress-intensity factor range tends to the fracture toughness of the material, K_{IC}, the Griffith-Irwin critical condition is reached and the curve presents a vertical asymptote. On the other hand, for very low values of ΔK_I, a pre-existing crack does not propagate and the curve presents another deviation from Paris' law. The so-called fatigue threshold, which was introduced by McClintok,[2] is defined in a conventional way as the value of SIF range below which a crack propagates at a growth rate less than 10^{-9} m/cycle. To take into account these deviations, Klesnil and Lukas[3] introduced the following relationship:

$$\frac{da}{dN} = \frac{C(\Delta K_I{}^m - \Delta K_{th}{}^m)}{(1-R)K_{IC} - \Delta K_I}, \tag{2}$$

which also considers the dependence of the crack growth rate on the loading ratio, R.

A few years after Paris' breakthrough, many researchers reported the observation that very short cracks are characterised by a crack growth rate higher than that experimentally determined for longer cracks,[4-6] the so-called anomalous behaviour of short cracks or small-crack problem.[7] More in detail, some authors have

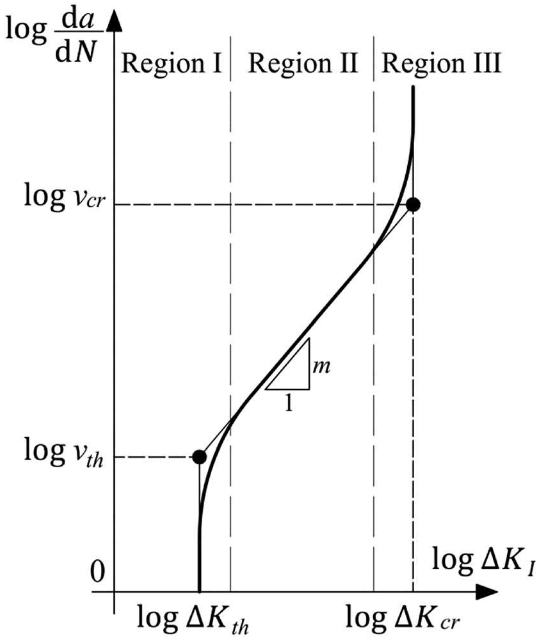

FIGURE 1 Sigmoidal or Paris' curve

proposed the subdivision of short cracks in three different typologies. The first one considers the microscopic short cracks, ie, when the crack length is small if compared with micro-structural dimensions, so that continuum mechanics breaks down. The second typology takes into account the mechanically small cracks. In this case, their length is small if compared with the dimension of the crack tip plastic zone and, therefore, the LEFM is no longer valid. Eventually, the physically small cracks are trivially referred to a crack length smaller than 0.5 to 1.0 mm.[6,8] Thus, if the data for long cracks are used to predict the lifetime of a component with small cracks, a dangerous overestimation of the minimum period of unrepaired service usage will be obtained leading to dramatic consequences. As a consequence, much effort has been made to understand the factors affecting the short crack growth behaviour. In particular, it has emerged that an important role in the so-called anomalous behaviour of small cracks has been played by the microstructural features. In fact, due to the interaction of the short fatigue crack with the grain boundaries, the crack path deflects from the macroscopic propagation direction. On the contrary, the presence of grain discontinuities has not influence on the propagation of long cracks, their mechanism being governed by average bulk properties, as stated in Suresh and Ritchie.[6] Furthermore, according to the above-mentioned definition of mechanically small cracks, a further contribution to different propagation mechanisms between longer and shorter cracks has been given by the local plasticity effect since the latter are characterised by a size comparable with crack tip plastic zone. Thus, the application of elastic-plastic fracture mechanics has been proposed in order to predict the small-crack advancement mechanism. In other words, the introduction of J-integral range or strain energy density variation as crack-driving force parameters may help to capture the anomalous short crack growth.[7] In addition, an important role in the explanation of short cracks propagation is given by the closure effect. In 1970, Elber first introduced a model based on crack closure concept in order to explain the effect of the loading ratio on crack growth rate.[9] In this approach, a physical contact between the fracture surfaces in the wake of crack tip was assumed. Therefore, the partial closure of the crack during the fatigue loading was taken into account considering that the crack growth rate can be related to the effective stress-intensity factor, which considers the various forms of crack closure, ie, plasticity ahead of the crack tip, roughness, and oxide-induced closure.[10,11] More in detail, the plasticity-induced closure can be due to the previously plastically stretched wake behind the crack tip, which comes into contact when the crack propagates in the plastic zone.[6] Therefore, the plastic wake behind the crack reduces the cyclic plastic deformations at the crack tip (for more details see previous studies[12,13]). Since small cracks are characterised by a limited wake in comparison to their longer counterparts, it can be supposed that the former ones will be less influenced by the closure effect.[6] In addition, it has been observed that the crack closure can be due also to the presence of corrosion debris within the crack itself or to the roughness of fracture surfaces. Moreover, some experimental results have highlighted that the crack closure increases with the crack length, thus it may be expected that long cracks propagate slower than shorter ones.[6] Eventually, it has been observed that chemical and electrochemical effects play an important role in the different propagation mechanisms between longer and shorter cracks. A possible explanation to this phenomenon has been given considering the differences in the crack tip environment, as well as to the influence of the crack length on the rate of renewal solution near the crack tip, as stated in Suresh and Ritchie.[6] More recently, a model that takes into account the elastic-plastic behaviour of short cracks near the crack tip was proposed for titanium alloys by Wang et al, who assumed a fictitious crack length according to Dugdale's model.[14] Subsequently, a modified UniGrow 2-parameter driving force model was proposed in order to predict the total fatigue life in presence of a short crack.[15,16] In addition, many researchers found that fatigue threshold is a crack-size-dependent parameter.[17] As a result, the fatigue threshold is a material constant only for sufficiently long cracks, whereas it decreases by decreasing the crack length, as is possible to note in

Kitagawa's diagram.[18,19] Notice that, since fatigue threshold represents a very important parameter in the fatigue problem, much of experimental and theoretical efforts were made during the last decades to understand this phenomenon.[20] For instance, Chapetti tried to provide an explanation to the crack-size dependence of the fatigue threshold by introducing the concept of micro-structural threshold for crack propagation.[21] The latter can be evaluated considering the fatigue limit of the material and the strongest micro-structural barrier. Hence, an effective fatigue threshold has been defined by the difference between the fatigue threshold for long cracks and the micro-structural threshold. Eventually, a comparison with some experimental data available in the literature has been done, which has shown a good agreement between the theoretical model proposed by Chapetti and the fatigue threshold data for all materials considered.

More recently, the concept of fractality has been exploited to explain the size effect on LEFM parameters.[22] Thus, based on the results obtained by Al. Carpinteri,[22] An. Carpinteri and Spagnoli proposed to apply the concepts of invasive fractal set in order to interpret the experimentally observed crack-size effect on Paris' regime and fatigue threshold.[23,24]

The purpose of this paper is to propose a re-examination of available models in literature and to obtain a unified treatise within the context of fractal geometry in order to clarify the crack-size effect on Paris' curve. In addition, the concepts of dimensional analysis and incomplete self-similarity to Paris' law are applied in order to determine a relationship that takes into account the crack size dependence of Paris parameter C.

Furthermore, by modelling the fracture surface as an invasive fractal set, it is possible to confirm the crack-size dependence of Paris' law. Analogously, scaling laws for the coordinates of the limit points are obtained in order to predict the crack-size effects on Paris' curve. Moreover, a multi-fractal scaling law (MFSL) is proposed for this parameter to model the variation of the fatigue threshold over a wide size range. Eventually, the proposed models are compared to experimental data available in the literature, which confirm the advantage of applying a fractal model to the short cracks problem.

2 | INTERMEDIATE ASYMPTOTICS THEORY IN THE ANALYSIS OF PARIS' LAW

Let us analyse the phenomenon of fatigue crack growth.[25-29] We assume the crack growth rate as the quantity to be determined. This quantity is influenced by the three different categories of variables. The first one takes into account the testing conditions, ie, the stress-intensity factor range, ΔK_I, and the loading ratio, R. The second category of parameters considers the static and cyclic material properties, such as the ultimate tensile stress, σ_u, the fracture toughness, K_{IC}, and the stress-intensity factor range threshold, ΔK_{th}. Eventually, the last category includes the geometric parameter, ie, the crack length, a.

Thus, we can write the following relationship:

$$\mathrm{d}a/\mathrm{d}N = \Pi(\Delta K_I, 1-R; K_{IC}, \sigma_u, \Delta K_{th}; a), \tag{3}$$

where we are neglecting the dependence on the time. The physical dimensions of the parameters entering Equation (3) are expressed in the Length-Force-Time class as follows:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = [\mathrm{L}], \Delta K_I = K_{IC} = \Delta K_{th} = [\mathrm{F}][\mathrm{L}]^{-3/2},$$
$$\sigma_u = [\mathrm{F}][\mathrm{L}]^{-2}, R = [-],\ a = [\mathrm{L}].$$

From dimensional analysis,[30] we select two dimensionally independent quantities, so that the number of parameters in Equation (3) can be reduced from six to four as follows:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{K_{IC}}{\sigma_u}\right)^2 \tilde{\Pi}\left(\frac{\Delta K_I}{K_{IC}}, 1-R; \frac{\Delta K_{th}}{K_{IC}}; \left(\frac{\sigma_u}{K_{IC}}\right)^2 a\right), \tag{4}$$

where the dimensionless parameters are as follows:

$$\Pi_1 = \frac{\Delta K_I}{K_{IC}}; \Pi_2 = 1-R;\ \Pi_3 = \frac{\Delta K_{th}}{K_{IC}};\ \Pi_4 = \left(\frac{\sigma_u}{K_{IC}}\right)^2 a.$$

Notice that Π_4 is responsible for the crack-size dependence of the fatigue crack growth. The application of Barenblatt-Botvina's approach allows us to further reduce the quantities involved in Equation (4). In fact, in Paris' law regime, the fatigue crack propagation mechanism does not depend on the initial and boundary conditions and, at the same time, the Griffith-Irwin critical condition has not been reached yet. As a consequence, Paris' law can be interpreted in the framework of intermediate asymptotics theory.[31] Hence, assuming an incomplete self-similarity in the corresponding dimensionless parameters, we obtain a power-law dependence of the crack growth rate on Π_i, with i =1,..,4[32-34] as follows:

$$\frac{\mathrm{d}a}{\mathrm{d}N}=\frac{(K_{IC})^{2-\alpha_1}}{\sigma_u{}^2}\Delta K_I{}^{\alpha_1}(1-R)^{\alpha_2}\left(\frac{\Delta K_{th}}{K_{IC}}\right)^{\alpha_3}\left(\frac{\sigma_u^2}{K_{IC}^2}a\right)^{\alpha_4}. \quad (5)$$

Comparing Equation (5) with Paris' law, we find the following relationships:

$$m=\alpha_1, \quad (6a)$$

$$C=\frac{(K_{IC})^{2-\alpha_1}}{\sigma_u{}^2}(1-R)^{\alpha_2}\left(\frac{\Delta K_{th}}{K_{IC}}\right)^{\alpha_3}\left(\frac{\sigma_u^2}{K_{IC}^2}a\right)^{\alpha_4}. \quad (6b)$$

Consequently, we can see how the parameter C represents a general power-law function of the crack length for a given material and loading condition. Therefore, Equation (6b) can be considered as a generalised Paris' law, in which all the main functional dependencies of the parameter C have been considered, allowing the detection of any anomalous deviations from the simple power-law regime first suggested by Paris.

It is interesting to note that, for $m = 2$, we obtain the following relationship:

$$\mathrm{d}a/\mathrm{d}N=(\Delta K_I/\sigma_u)^2, \quad (7)$$

which corresponds to the so-called complete self-similarity, as pointed out by Barenblatt and Botvina.[25]

3 | SOME REMARKS ON THE INVASIVE FRACTAL SCALING LAWS

It is well known that a similar morphology of the fracture surface appears over a wide range of observation scales.[35] In other terms, the fracture surface can be modelled as a fractal set of dimension greater than two, where its non-integer dimension is the way to quantify the disorder.

Thus, supposing that the total energy dissipation necessary to break a specimen occurs in a fractal domain of dimension $\alpha=2+d_{\mathcal{G}}$, with $0\le d_{\mathcal{G}}\le 1$, the application of the renormalization group theory provides the following scaling law for the fracture energy[22,35,36]:

$$\mathcal{G}_F\simeq\mathcal{G}_F^*\,b^{d_{\mathcal{G}}}, \quad (8)$$

where $\mathcal{G}_F^*$ represents the scale-invariant fracture energy with anomalous physical dimensions $[\mathrm{F}][\mathrm{L}]^{-(1+d_{\mathcal{G}})}$, which is intermediate between a classical fracture energy and a plastic bulk dissipation.[37-39]

Let us consider now an infinite plate with a fractal crack of projected length $2a$, where the invasive fractal dimension of the crack profile is equal to $1+d_{\mathcal{G}}$, so that the fractal measure of the crack is

$$a^*\simeq a^{1+d_{\mathcal{G}}}. \quad (9)$$

Considering the Irwin's relationship for the critical values and making the following position:

$$K_{IC}^*\simeq\sqrt{\mathcal{G}_F^*E}, \quad (10)$$

Equation (8) becomes:

$$K_{IC}\simeq K_{IC}^*\,a^{\frac{d_{\mathcal{G}}}{2}}, \quad (11)$$

which, generalised to non-critical values, allows us to obtain[22]

$$K_I\simeq K_I^*a^{\frac{d_{\mathcal{G}}}{2}}, \quad (12)$$

with:

$$\left[K_I^*\right]=[\mathrm{F}][\mathrm{L}]^{-(3+d_{\mathcal{G}})/2}. \quad (13)$$

It is interesting that, for $d_{\mathcal{G}}\to 1$, ie, for an energy dissipation in the volume, K_I^* will have the same physical dimensions of stress so that the stress singularity vanishes.

This approach is strictly valid only within a limited scale range. In fact, as the size increases, the concept of geometrical multi-fractality implies the progressive vanishing of fractality, ie, there is a change of the fractal dimension with the scale of observation.[40,41]

In other words, since the microstructure of a disordered material remains the same independently of the scale of observation, the influence of disorder on mechanical properties depends on the ratio between a characteristic material length, l_{ch}, and the external size of the specimen, b. Hence, the effect of microstructural disorder on the mechanical behaviour of the material becomes progressively less important at the larger scales and, eventually, fractality vanishes for b tending to infinity.

This transition from a disordered regime for smaller scales, where a Brownian disorder holds with a fractal scaling exponent equal to 1/2, to an ordered regime for larger scales can, therefore, be considered in the scaling of any mechanical quantity. The analytical expression of the multi-fractal scaling law (MFSL) for fracture energy is shown in Figure 2[41]:

$$\mathcal{G}_F = \mathcal{G}_F^{\infty}\left(1+\frac{l_{ch}}{b}\right)^{-1/2}. \tag{14}$$

This scaling law is a two-parameter best-fitting, where the asymptotic value of the nominal quantity, corresponding to the highest value of the fracture energy, is reached only in the limit case of infinite specimen sizes. The dimensionless term within round brackets represents the variable influence of disorder on the mechanical behaviour by means of the characteristic length, l_{ch}. On the other hand, in the bi-logarithmic diagram, the transition from the fractal scaling regime to the Euclidian one is represented by the point of abscissa $\log l_{ch}$.[42-44]

4 | FRACTAL AND MULTI-FRACTAL APPROACH TO PARIS REGIME

In the most recent decades, experimental evidence has shown a dependence of crack growth rate on crack-size, the stress-intensity factor range being the same. Although different approaches have been introduced to describe the anomalous behaviour of short cracks, a rigorous explanation to the short cracks problem can be given by the concepts of fractal geometry. In fact, the latter is able to capture the short crack propagation behaviour by taking into account directly the deflections of the crack path from the macroscopic propagation direction. In other words, the fractal geometry allows to consider the positive effect of tortuosity in crack path by modelling the crack profile as an invasive fractal set, without the need to consider explicitly the mechanism of crack deflection in the propagation of short fatigue cracks, which is due to the interaction with the grain boundaries and microstructural discontinuities. On the other hand, it has been observed that the plasticity-induced crack closure model is not able to predict correctly the anomalous short crack behaviour as stated in Vasudeven et al.[45] The fractal geometry was first applied by Mandelbrot et al.[46] to characterise the fractal nature of fracture surfaces of metals and, in the last few years, noticeable advances have been made in the study of the fractal aspects of crack morphology and energy dissipation over fractal domains. Thus, considering the renormalized quantities related to the fractal crack represented in Equations (9) and (12) in fatigue crack growth law, a crack-size dependent Paris' law can be found. By employing Equation (12), the nominal stress-intensity factor range, ΔK_I, can be related to the corresponding fractal parameter, ΔK_I^*, by the following relationship:

$$\Delta K_I \simeq \Delta K_I^* a^{\frac{d_{\mathcal{G}}}{2}}. \tag{15}$$

Furthermore, by applying the derivation rule for composite functions to the nominal crack propagation rate, it is possible to express the latter as a function of the fractal crack propagation rate $\mathrm{d}a^*/\mathrm{d}N$:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \frac{\mathrm{d}a}{\mathrm{d}a^*}\frac{\mathrm{d}a^*}{\mathrm{d}N}, \tag{16}$$

where

$$\frac{\mathrm{d}a}{\mathrm{d}a^*} = \frac{a^{-d_{\mathcal{G}}}}{(1+d_{\mathcal{G}})}. \tag{17}$$

Thus, the nominal crack propagation rate in Equation (16) is given by:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \frac{\mathrm{d}a^*}{\mathrm{d}N}\frac{a^{-d_{\mathcal{G}}}}{1+d_{\mathcal{G}}}, \tag{18}$$

which, substituted into Paris' law together with Equation (15), allows us to obtain:

$$\frac{\mathrm{d}a^*}{\mathrm{d}N}\frac{a^{-d_{\mathcal{G}}}}{1+d_{\mathcal{G}}} = C\,\Delta K_I^{*m} a^{\frac{m}{2}d_{\mathcal{G}}}. \tag{19}$$

Now, the following position can be put forward:

$$\mathrm{d}a^*/\mathrm{d}N = C^*\,\Delta K_I^{*m}. \tag{20}$$

In other terms, Equation (20) introduces the fractal Paris' parameter C^*, which is a crack-size-independent

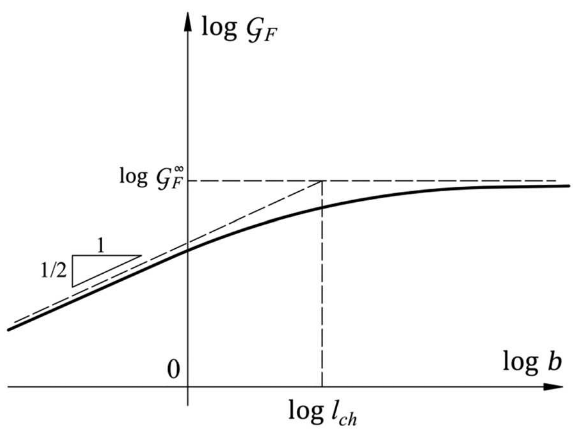

FIGURE 2 Multi-fractal scaling law for fracture energy

parameter. Finally, this relationship, once substituted into Equation (19), allows us to find

$$C^* = (1+d_{\mathcal{G}})\,C\,a^{d_{\mathcal{G}}\left(1+\frac{m}{2}\right)}. \tag{21}$$

Hence, inverting the previous formula, the scaling law for Paris' parameter C is obtained as follows:

$$C(a) = \frac{C^*}{1+d_{\mathcal{G}}}a^{-d_{\mathcal{G}}\left(1+\frac{m}{2}\right)}. \tag{22}$$

In this way, Equation (22) leads to a crack-size dependence of crack propagation law as follows:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \frac{C^*}{1+d_{\mathcal{G}}}a^{-d_{\mathcal{G}}\left(1+\frac{m}{2}\right)}\Delta K_I^m, \tag{23}$$

which can be regarded as a modified Paris' law, since the parameter C is no longer a material constant (see Figure 3A). In other words, Equation (23) predicts a decrease in the crack growth rate, $\mathrm{d}a/\mathrm{d}N$, by increasing the crack-size (see Figure 3B). Thus, comparing Equation (22) with Equation (6b), we note that the assumption of the invasive fractal roughness of crack profile implies an incomplete self-similarity in the problem.

It is interesting to specify that Equation (23) was obtained by Andrea Carpinteri and Spagnoli,[23,24] but following the diametrically opposite path. In fact, in the present paper, Equation (8) has been found herein starting from the well-known Paris' law. Thus, exploiting the relationships between nominal quantities and corresponding fractal ones, Equation (8) has been directly demonstrated. Although the two approaches permit us to obtain the same scaling law for parameter C, nevertheless the modus operandi herein proposed is logically more consistent.

In addition, it is interesting to note that the fractal Paris' law in Equation (23) and the NASGRO crack growth equation can capture equally well the anomalous small crack behaviour. In fact, as stated in Jones et al,[47] the NASGRO crack growth equation can be used to represent the small crack propagation if closure effects are neglected and the fatigue threshold is set to a very small value. Furthermore, notice that the scaling law proposed for Paris parameter C can be applied only to a limited crack-size range in order to consider a constant value of the dimensional increment $d_{\mathcal{G}}$. Otherwise, although the general trend can be captured considering just the fractal approach, a transition occurs from a fractal regime for small cracks to a Euclidian regime for long cracks.

Exploiting the concept of self-affinity, the fractal increment with respect to 2 can be assumed to vary with the scale of observation so that it becomes a function of the crack length. Therefore, according to Equation (22), a MFSL for Paris' parameter C can be put forward[48] as follows:

$$C_{MF}(a) = C_{MF}^{\infty}\left(1+\frac{l_{ch}}{a}\right)^{d_{\mathcal{G}}\left(1+\frac{m}{2}\right)}. \tag{24}$$

This relationship is shown in Figure 4 and connects the two asymptotic behaviours for shorter and longer cracks. For shorter cracks, the maximum possible disorder is reached and an oblique asymptote, with slope equal to $-d_{\mathcal{G}}(1+m/2)$, is obtained. On the other hand, for very long cracks, the dependence on the crack-size disappears and a horizontal asymptote is found, ie, $d_{\mathcal{G}} \to 0$. Additionally, notice that l_{ch} represents the crack length that separates the two asymptotic behaviours. Eventually, we can write the following modified Paris' law:

$$\mathrm{d}a/\mathrm{d}N = C_{MF}(a)\,\Delta K_I{}^m, \tag{25}$$

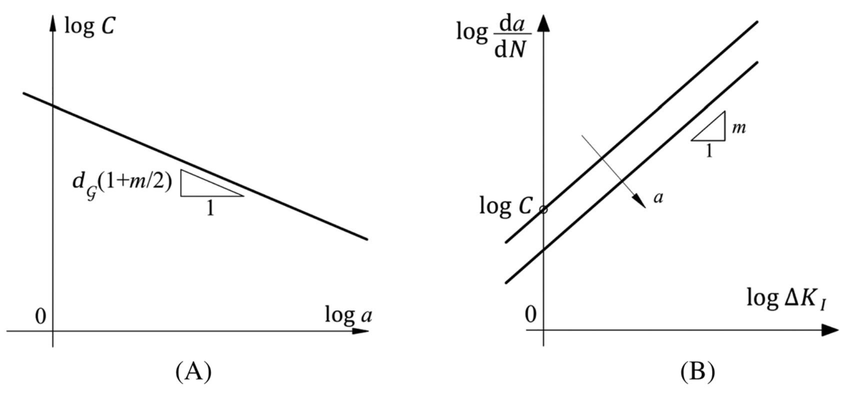

FIGURE 3 A, Crack-size effect on Paris' parameter C. B, Crack-size effect on Paris' law

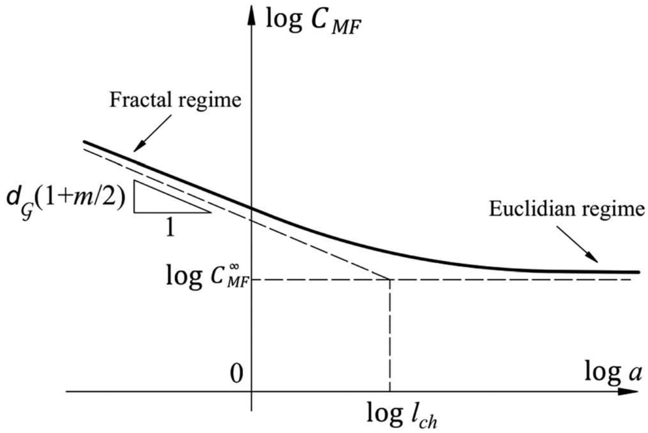

FIGURE 4 Multi-fractal scaling law for Paris' parameter C

where the expression of $C_{MF}(a)$ is given by Equation (25). Eventually, note that the aforementioned decrease in the dimensional increment $d_{\mathcal{G}}$ with the scale of observation is consistent with the experimental evidence that the fractal dimension changes when the segment of profile measurement approaches the size of the structural element, since the effect of the microstructural disorder on the mechanical behaviour becomes progressively less important at a larger scale.

5 | CORRELATION BETWEEN BARENBLATT-BOTVINA'S AND FRACTAL APPROACHES TO PARIS' REGIME

The purpose of the present section is to find a correlation between the two approaches previously proposed so that a relationship can be determined between the dimensional increment of fractal approach, $d_{\mathcal{G}}$, and m. By considering the fractal crack in Equation (9), Paris' law can be written in the following form:

$$\frac{\mathrm{d}a^{1+d_{\mathcal{G}}}}{\mathrm{d}N} \simeq \left(\frac{\Delta K_I}{\sigma_u}\right)^m, \tag{26}$$

which, from a dimensional point of view, implies:

$$[\mathrm{L}]^{1+d_{\mathcal{G}}} = [\mathrm{L}]^{m/2}. \tag{27}$$

Equation (27) shows that, for $m = 2$, the dimensional increment is equal to zero so that a flat crack is obtained. In other words, for $m = 2$, the crack-size effect on Paris' law disappears, which corresponds to the case of complete self-similarity shown in Equation (7):

$$m = 2, \tag{28a}$$

$$\frac{\mathrm{d}a^1}{\mathrm{d}N} \simeq \left(\frac{\Delta K_I}{\sigma_u}\right)^2, \tag{28b}$$

$$d_{\mathcal{G}} = 0. \tag{28c}$$

Let us consider the case of $m = 3$. Following the same path of reasoning, Equation (27) implies a value of the dimensional increment equal to 1/2, which leads to the concept of Brownian surface in the context of fractal geometry:

$$m = 3, \tag{29a}$$

$$\frac{\mathrm{d}a^{3/2}}{\mathrm{d}N} \simeq \left(\frac{\Delta K_I}{\sigma_u}\right)^3, \tag{29b}$$

$$d_{\mathcal{G}} = 1/2. \tag{29c}$$

As a result, Equation (25) allows us to demonstrate that the assumption of fractal roughness in the crack profile implies the incomplete self-similarity, being $m > 2$. It is interesting to note that Equation (29b) can be seen as a generalisation of Frost-Dugdale crack growth law.[49] In fact, Jones et al obtained the following generalised Frost-Dugdale law[50-53]:

$$\frac{\mathrm{d}a}{\mathrm{d}N} \simeq \left(\frac{\Delta K_I^{1-p} K_{max}^p}{\sigma_u}\right)^m a^{1-\frac{m}{2}}. \tag{30}$$

Furthermore, by considering the relationship between the stress-intensity factor range and the loading ratio, the previous equation can also be written in the following form:

$$\frac{\mathrm{d}a}{\mathrm{d}N} \simeq \left(\frac{\Delta K_I (1-R)^{-p}}{\sigma_u}\right)^m a^{1-\frac{m}{2}}, \tag{31}$$

which was confirmed by experimental results.[48] For instance, the material constants obtained for AA7050-T7451, $p = 0.2$ and $m = 3$, yield to the following expression:

$$\mathrm{d}a/\mathrm{d}N \simeq \left(\frac{\Delta K_I}{\sigma_u}\right)^3 a^{-1/2}, \tag{32}$$

in agreement with Equation (29b). Moreover, noteworthy is the Williford's paper[54] in which he modelled fracture surface at the crack tip as an invasive fractal and a power-law was considered for J integral. Williford obtained a crack-size dependent Paris' law in which both parameter m and exponent of crack length are functions of the fractal dimension. It is very interesting to note that Williford's model provides $m = 3$ for $d_{\mathcal{G}} = 0.5$, in perfect agreement with Equation (29b) and, in particular, with experimental data. For instance, for AISI 4340 steel, the measured value of the fractal dimension under monotonic loading is equal to 1.51,[55] as well as $m = 2.82$, which is very close to 3.[56] Eventually, we should analyse the case of $m = 4$. In fact, by inserting this value into Equation (26), we obtain

$$m = 4, \tag{33a}$$

$$\frac{\mathrm{d}a^2}{\mathrm{d}N} \simeq \left(\frac{\Delta K_I}{\sigma_u}\right)^4, \tag{33b}$$

$$d_{\mathcal{G}} = 1, \tag{33c}$$

which corresponds to the maximum value for $d_{\mathcal{G}}$ and to maximum disorder and volumetric plastic dissipation. Thus, we can summarise as follows:

$$0 \le d_{\mathcal{G}} \le 1, \tag{34a}$$

$$2 \le m \le 4. \tag{34b}$$

In other words, Equation (34b) allows us to say that values of m higher than 4 are not permitted, otherwise we would have the paradox of a planar dimension larger than 2. On the other hand, experimentally determined values of m higher than 4 have been observed for brittle materials, like concrete or ceramics. Therefore, it seems reasonable to suppose that the approach here proposed can be applied to the analysis of ductile fatigue fractures, which are characterised by values of m lower than 4.

6 | FRACTAL AND MULTI-FRACTAL APPROACH TO PARIS' CURVE

In this section, we propose the fractal approach for the study of crack-size effects on coordinates of the limit-points of Paris' curve. To this aim, Equations (12) and (17) are applied to the coordinates of fatigue threshold point in order to obtain the following scaling laws for ΔK_{th} and v_{th}:

$$\Delta K_{th} \simeq \Delta K^*_{th}\, a^{\frac{d_{\mathcal{G}}}{2}}, \tag{35a}$$

$$v_{th} \simeq v^*_{th}\, a^{-d_{\mathcal{G}}}. \tag{35b}$$

Thus, Equation (35a) implies that, in accordance with the hypothesis of fractal roughness of the crack profile, the fatigue threshold increases with the crack length, whose trend is confirmed by the experimental data. Also, notice that, for $d_{\mathcal{G}} \to 1$, we obtain $\Delta K_{th} \propto \sqrt{a}$, which implies that the fractal fatigue threshold assumes the physical dimensions of stress or, in other words, that the energy dissipation occurs in the bulk volume.[57] On the other hand, Equation (35b) predicts a decrease in threshold growth rate, v_{th}, with the crack length.

Analogously, evaluating Equations (12) and (17) in correspondence of the coordinates of the critical point of Paris' curve, the following scaling laws can be put forward:

$$\Delta K_{cr} \simeq (1-R) K^*_{IC}\, a^{\frac{d_{\mathcal{G}}}{2}}, \tag{36a}$$

$$v_{cr} \simeq v^*_{cr}\, a^{-d_{\mathcal{G}}}, \tag{36b}$$

which predict the same trend given by Equations (35a) and (35b). Therefore, according to Equations (35a) and (36a), an increase in fatigue threshold and fracture toughness is expected with the crack length, which can be ascribed to the fractality of rough crack. Moreover, notice that ΔK^*_{th} and ΔK^*_{cr} are the corresponding fractal quantities of fatigue threshold and critical SIF range, which can be seen as material constants with anomalous dimensions. On the other hand, Equations (35b) and (36b) predict a negative scaling for v_{th} and v_{cr}, ie, these parameters decrease with the crack length.

As a result, the scaling laws previously introduced yield a simultaneous rightward and downward translation of Paris' curve increasing the crack length, ie, we obtain a crack-size-dependent Paris' curve, as shown in Figure 5A.

In addition, substituting the nominal crack growth rate and the nominal SIF-range with the corresponding renormalized parameters, a fractal-coordinate system is obtained. Consequently, through the introduction of fractal coordinates, the set of Paris' curves, obtained varying the crack length, collapse onto a single fractal (crack-size independent) Paris' curve, see Figure 5B. In other words, the application of renormalization group theory leads to a fractal diagram where the coordinates of the limit-points of Paris' curve correspond to the fractal quantities entering Equations (35a,b) and (36a,b).

Furthermore, analogously to what has been done in Section 4, a multi-fractal approach for the fatigue

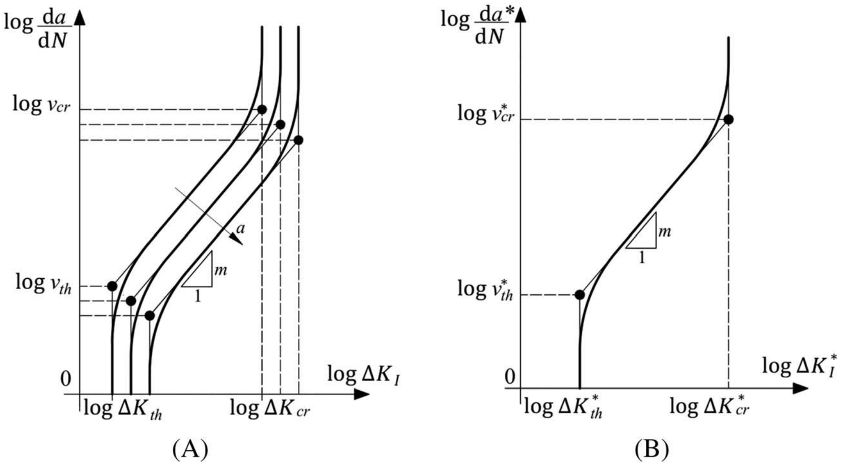

FIGURE 5 A, Crack-size-dependent Paris' curve. B, Fractal (crack-size independent) Paris' curve

threshold should be considered in order to link the two extreme behaviours so that the experimental trend is reproduced.

In fact, according to Kitagawa's diagram, a transition of fatigue threshold occurs from the long cracks regime, where the fatigue threshold is a material property, to the short cracks regime.[58,59] Therefore, since the use of a multi-fractal scaling law is strictly needed to model the variation of the fatigue threshold over a wide size range, the following scaling law should be put forward (Figure 6):

$$\Delta K_{th} = \Delta K^{\infty}_{th}\left(1+\frac{l_{ch}}{a}\right)^{-\frac{d_G}{2}}. \tag{37}$$

According to Equation (37), the fatigue threshold increases with the crack length until, for very long cracks, the fractality vanishes and the highest value for the fatigue threshold is achieved, ΔK^{∞}_{th}. On the other hand, for very small cracks, the influence of the material

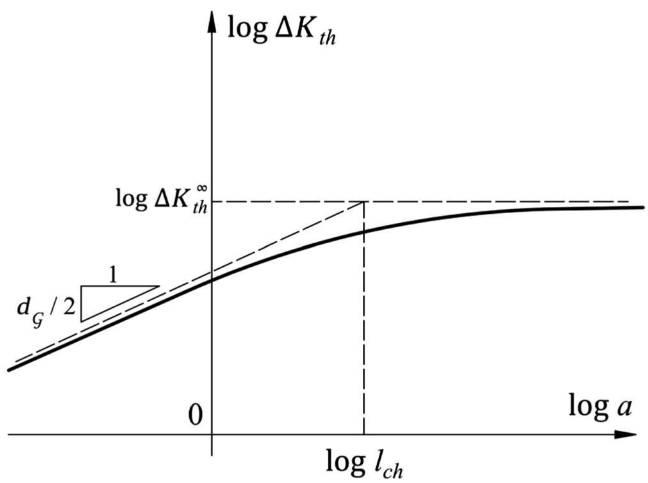

FIGURE 6 Multi-fractal scaling law for fatigue threshold

disorder becomes progressively more important and the fatigue threshold tends to vanish. Notice that, for very small cracks, Equation (37) provides an oblique asymptote in the bi-logarithmic diagram with slope equal to the exponent of the term within round brackets, which corresponds to the maximum possible disorder for a given material. On the other hand, for very long cracks, the influence of disorder on the fatigue behaviour tends to vanish, so that crack-size dependence disappears and we obtain the asymptotic value of the fatigue threshold, ΔK^{∞}_{th}, which can be seen as a material constant.

Furthermore, it should be pointed out that, although Equation (37) predicts the same trend of that obtained by Carpinteri and Paggi[58] for the fatigue threshold, it shows as the scaling exponent only depends on the fractal increment, d_G, which takes into account the microstructural disorder of the material. Conversely, in Carpinteri and Paggi,[59] the scaling exponent related to fatigue threshold is function of both fractal increment, d_G, and Paris' parameter, m, which instead governs the fatigue crack propagation phenomenon only in Region II.

Further investigations are needed to relate l_{ch} with the fatigue threshold crack length, $\bar{a}_0$, where the former is the transition length relating the two extreme asymptotic behaviours, whereas the latter is the length of the micro-defect above which a subcritical crack propagation may occur[60]:

$$\bar{a}_0 = \frac{1}{\pi}\left(\frac{\Delta K^{\infty}_{th}}{\Delta \sigma_{fl}}\right)^2, \tag{38}$$

where ΔK^{∞}_{th} refers to the fatigue threshold for very long cracks, whereas $\Delta \sigma_{fl}$ represents the fatigue limit. Eventually, notice that the fatigue threshold crack length, $\bar{a}_0$, shows a structure similar to that of the characteristic crack length, a_0. The latter represents the minimum crack-size related to Griffith-Irwin critical condition and is given by the following relationship:

$$a_0 = \frac{1}{\pi}\left(\frac{K_{IC}}{\sigma_u}\right)^2. \tag{39}$$

7 | EXPERIMENTAL RESULTS ON PARIS' LAW AND FATIGUE THRESHOLD

In this section, we consider some experimental data obtained by investigating Ni-based super-alloy samples with different initial crack sizes, which are subjected to four-point fatigue bending tests.[61] It must be taken into consideration that a proper comparison should require geometrical self-similar specimens. Two different Paris' curves in the midrange of growth rates are obtained for different initial crack lengths, as shown in Figure 7A. When experimental results are reported in the fractal Paris diagram, they collapse onto a single straight line, independently of the initial crack length. Thus, the renormalization of the Paris' curve yields to a scale-invariant curve. Best fitting of the experimental data collected in Luo and Bowen,[61] in order to collapse them onto a single curve, provides the dimensional increment $d_\mathcal{G}$, the fractal Paris' parameter C, and the exponent of Paris' law m, which assume the values reported in Table 1. Eventually, the experimental data are reported in the fractal Paris' diagram, where they all collapse along a single straight line (Figure 7B).

In addition, we consider the experimental results on fatigue threshold available in literature. By performing a nonlinear regression analysis on the experimental data,[18,19,60,62,63] an experimental assessment of Equation (37) is proposed in Figure 8 through which we evaluate the free parameters entering Equation (37), ie, the asymptotic value of the fatigue threshold, ΔK_{th}^{∞}, the fractal increment, $d_\mathcal{G}$, and the characteristic length of the material, l_{ch}. Thus, the analysis of experimental data

TABLE 1 Fitting parameters of Equation (23)

m	C (mm$^{6.40}$ N$^{-2.97}$)	$d_\mathcal{G}$
2.97	3.125 10^{-14}	0.38

shows how the dimensional increment, $d_\mathcal{G}$, varies between 1/2, in the case of Brownian disorder, and 1, in the case of a volumetric bulk dissipation, whereas the transition length, l_{ch}, ranges from some micrometres up to approximately 1 mm. In addition, it is interesting to note that the experimental values of l_{ch} obtained from the best-fitting procedure can be connected to the grain size of the material. In fact, in Section 3, it has been affirmed that the characteristic material length, l_{ch}, has been connected to the effect of micro-structural disorder of the material. Since it is reasonable to assume that the micro-structural disorder can depend on the grain size, therefore it follows that the horizontal coordinate of the intersection point of the two asymptotes, l_{ch}, can be related to the grain size of the material. To this aim, it is interesting also to consider the best-fitting parameters of the experimental data collected in Tanaka et al.,[62] which are reported in Figures 8c and 8d. For the material with the most ordered microstructure, ie, the ferritic steel with grain size equal to 7.8 μm, the characteristic material length, l_{ch}, is smaller than the one obtained for the pearlitic steel, which is characterised by a grain size of 55 μm. Analogously, Figure 8F shows as FC20 cast iron, which represents the most disordered material considered in the present paper, is characterised by a higher value of l_{ch}. In fact, as stated in Usami,[63] the grain size of this cast iron can increase by some millimetres. Thus, the application of MFSL to the experimental data of FC20 cast iron provides a value of the characteristic material length that is in agreement with its grain size, the former being equal to 2,017 μm. The characteristic material length, l_{ch}, can be also related to the intrinsic crack size, which has first been proposed by El Haddad et al.[60] In fact, if the experimental data reported in Figure 8E are taken into consideration, it is remarkable to highlight that the best-fitting parameter, l_{ch}, has the same order of magnitude of

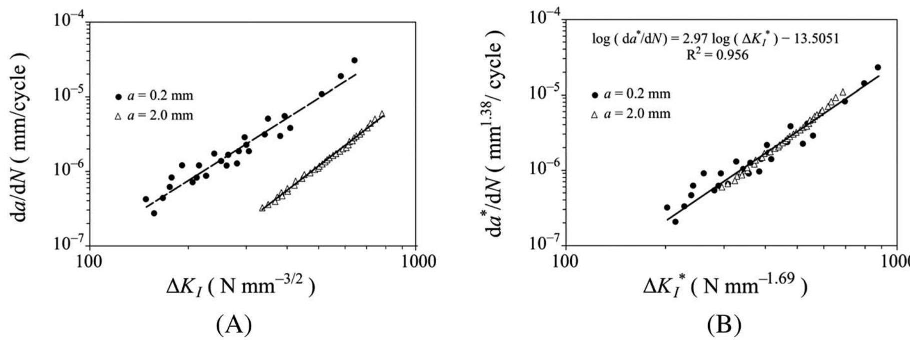

FIGURE 7 A, Experimental crack-size-dependent Paris' curves in the power-law regime. B, Experimental fractal Paris' curve in the power-law regime

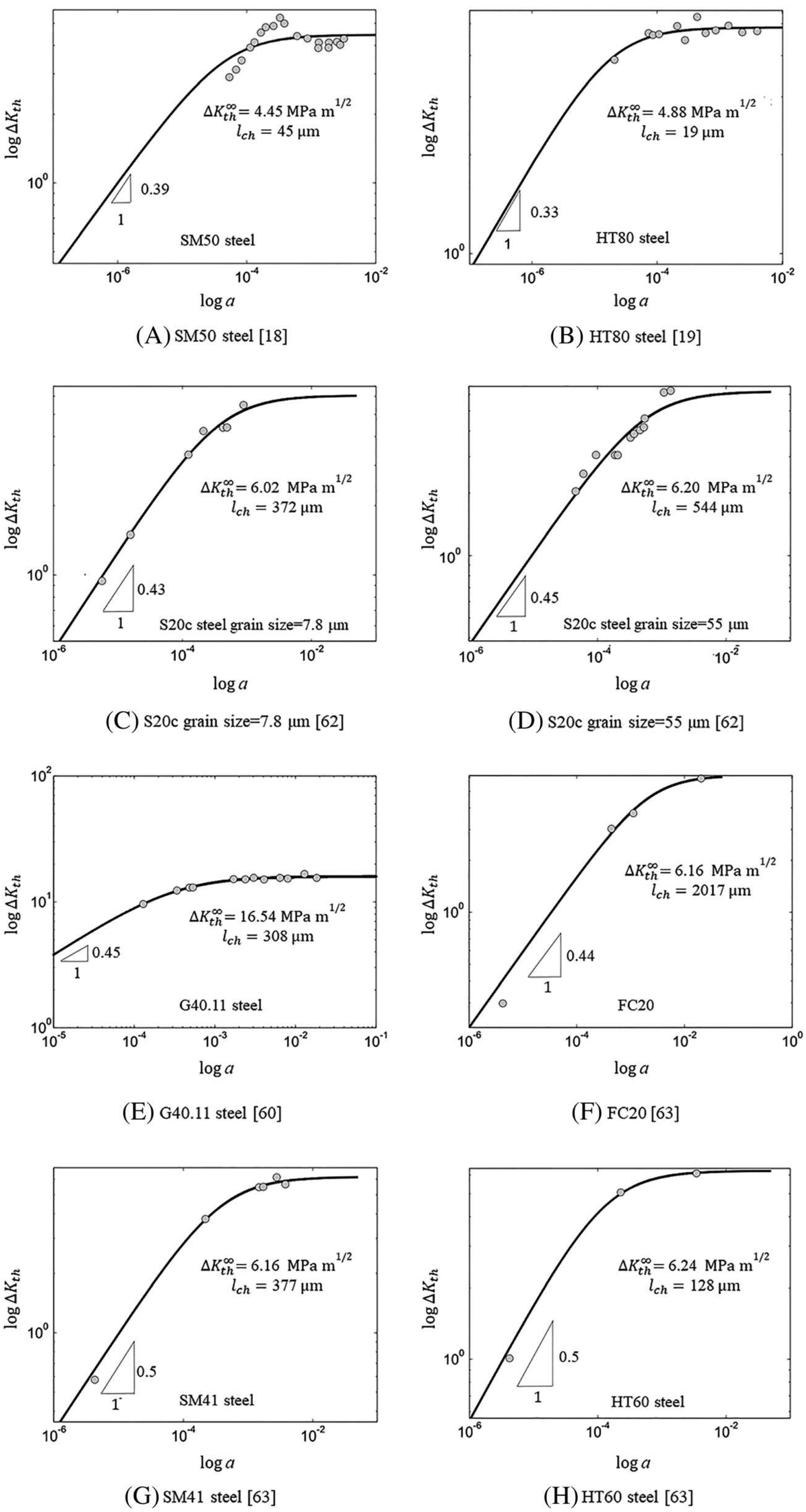

FIGURE 8 Experimental assessment of multi-fractal scaling law for fatigue threshold. ΔK_{th} is measured in (MPa m$^{1/2}$), whereas l_{ch} is measured in (m)

the parameter obtained by El Haddad et al. in a previous study.[60] More in detail, the parameter entering the MFSL of Equation (37) is equal to 308 μm, whereas the estimated value for the intrinsic crack size is 240 μm.

Eventually, considering the fatigue property chart reported in Fleck et al,[64] it is interesting to evaluate the orders of magnitude between the fatigue threshold crack length, $\bar{a}_0$, and the minimum crack-size related to Griffith-Irwin critical condition, a_0. To this aim, we consider the orders of magnitude of the parameters entering Equations (38) and (39). Starting with the fracture toughness and fatigue threshold, the experimental data allow us to say that two orders of magnitude separate these parameters as follows:

$$\Delta K_{th}^{\infty} \cong 10^{-2} K_{IC}. \tag{40}$$

On the other hand, only one order of magnitude separates ultimate strength and fatigue limit:

$$\Delta \sigma_{fl} \cong 10^{-1} \sigma_u. \tag{41}$$

Thus, substituting these relationships into Equations (38) and (39), we can assert that about two orders of magnitude separate fatigue threshold crack length and transitional crack length:

$$\bar{a}_0 \cong 10^{-2} a_0, \tag{42}$$

which is in agreement with the experimental data.

8 | CONCLUSIONS

When Paris' law was introduced in 1963, size effects were not yet known. In fact, the so-called anomalous behaviour of short cracks has only been investigated from the early 1980s through the concepts of plasticity-induced crack closure, roughness-induced crack closure, or the use of empirical laws. Only in more recent times, through the use of self-similarity concepts and fractal geometry, it has been possible to give a consistent theoretical explanation to the size-scale problem. In this paper, the application of dimensional analysis and the intermediate asymptotics theory are used to find generalised formulations for Paris' law. Furthermore, by proposing a modelling based on the concept of the invasive fractal crack, a crack-size dependence of Paris' law is introduced so that a correlation between the intermediate asymptotics theory and fractal geometry is found. Subsequently, a multi-fractal scaling law for fatigue threshold is proposed to find a correlation with the crack length in accordance with Kitagawa's diagram. The use of the fractal and multi-fractal approaches can explain the increment in fatigue threshold with the crack length. Eventually, the model proposed in this paper is compared very satisfactorily with experimental data available in the literature.

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NOMENCLATURE

$\Delta\sigma_{fl}$	fatigue limit
ΔK_I	stress-intensity factor range (SIF range)
ΔK_I^*	fractal stress-intensity factor range
ΔK_{cr}	critical stress-intensity factor range
ΔK_{cr}^*	fractal critical SIF range
ΔK_{th}	fatigue threshold
ΔK_{th}^*	fractal fatigue threshold
ΔK_{th}^{∞}	asymptotic value of fatigue threshold
σ_u	ultimate tensile strength
a	crack length
$\bar{a}_0$	fatigue threshold crack length
a_0	micro-defects characteristic length
b	specimen size
C	Paris' parameter
C^*	fractal Paris' parameter
C_{MF}	asymptotic value of Paris parameter
$\mathrm{d}a/\mathrm{d}N$	fatigue crack propagation rate
$\mathrm{d}a^*/\mathrm{d}N$	fractal fatigue crack propagation rate
$d_{\mathcal{G}}$	dimensional increment
E	Young's modulus
$\mathcal{G}_F$	fracture energy
$\mathcal{G}_F^*$	fractal fracture energy
$\mathcal{G}_F^{\infty}$	asymptotic value of fracture energy
K_{IC}	fracture toughness
K_{IC}^*	fractal fracture toughness
K_{Imax}	maximum value of stress-intensity factor
l_{ch}	characteristic material length
m	Paris' exponent
p	Frost-Dugdale's law parameter
R	loading ratio
v_{cr}	critical value of the fatigue crack propagation rate
v_{cr}^*	fractal critical value of the fatigue crack propagation rate
v_{th}	threshold value of the fatigue crack propagation rate
v_{th}^*	fractal threshold value of the fatigue crack propagation rate

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REFERENCES

1. Paris P, Erdogan F. A critical analysis of crack propagation laws. *Journal of Basic Engineering Transaction ASME*. 1963;85:528-534.

2. McClintock FA. On the plasticity of the growth of fatigue cracks. In: *Fracture of Solids*. New York: John Wiley & Sons Inc.; 1963:65.
3. Klesnil M, Lukas P. Effect of stress cycle asymmetry on fatigue crack growth. *Mater Sci Eng A*. 1972;9:231-240.
4. Pearson S. Initiation of fatigue cracks in commercial aluminium alloys and the subsequent propagation of very short cracks. *Engineering Fracture Mechanics*. 1975;7:241-247.
5. Miller KJ. The short crack problem. *Fatigue Fract Eng Mater Struct*. 1982;5:223-232.
6. Suresh S, Ritchie RO. Propagation of short fatigue cracks. *International Metals Reviews*. 1984;29:445-475.
7. Ritchie RO, Lankford J. Small fatigue cracks: A statement of the problem and potential solutions. *Mater Sci Eng A*. 1986;84:11-16.
8. Pugno N, Ciavarella M, Cornetti P, Carpinteri A. A generalized Paris' law for fatigue crack growth. *J Mech Phys Solids*. 2006;54:1333-1349.
9. Elber W. Fatigue crack closure under cyclic tension. *Engineering Fracture Mechanics*. 1970;2:37-44.
10. Suresh S, Ritchie RO. Near-threshold fatigue crack propagation: a perspective on the role of crack closure. In: Davidson DL, Suresh S, eds. *Proceedings of Fatigue Crack Growth Threshold Concepts*. Philadelphia: Metallurgical Society of AIME; 1984:227-261.
11. Taylor D. Fatigue of short cracks: the limitations of fracture mechanics. In: Miller KJ, de los Rios ER, eds. *The Behaviour of Short Fatigue Cracks*. London, Mechanical Engineering Publications; 1986:479-490.
12. Pippan R, Hohenwarter A. Fatigue crack closure: a review of the physical phenomena. *Fatigue Fract Eng Mater Struct*. 2017;40:471-495.
13. Zerbst U, Vormawald M, Pippan R, Gänser HP, Baudoux CS, Madia M. About the fatigue crack propagation threshold of metals as a design criterion – A review. *Engineering Fracture Mechanics*. 2016;153:190-243.
14. Wang K, Wang F, Cui W, Hayat T, Ahmad B. Prediction of short fatigue crack growth of Ti-6Al-4 V. *Fatigue Fract Eng Mater Struct*. 2014;37:1075-1086.
15. Bang DJ, Ince A, Tang LQ. A modification of UniGrow 2-parameter driving force model for short fatigue crack growth. *Fatigue Fract Eng Mater Struct*. 2018;42:45-60.
16. Bang DJ, Ince A, Noban M. Modeling approach for a unified crack growth model in short and long fatigue crack regimes. *International Journal of Fatigue*. 2019;128:45-60.
17. Frost NE. The growth of fatigue cracks. In: Yokobori T, ed. *Proceedings of the First International Conference on Fracture*. Sendai: The Japan Society for Strength and Fracture of Materials; 1966:1433-1459.
18. Kitagawa H, Takahashi S. Applicability of fracture mechanics to very small cracks or the cracks in the early stage. In: *Proceedings of the Second International Conference on Mechanical Behavior of Materials*. Boston, Massachusetts: American Society for Metals; 1976:627-631.
19. Kitagawa H, Takahashi S. Fracture mechanical approach to very small fatigue cracks and to the threshold condition. *Transaction of the Japan Society of Mechanical Engineers*. 1979;45:1289-1303.
20. Maierhofer J, Pippan R, Gänser HP. Modified NASGRO equation for physically short cracks. *International Journal of Fatigue*. 2014;59:200-207.
21. Chapetti MD. Fatigue propagation threshold of short cracks under constant amplitude loading. *International Journal of Fatigue*. 2003;25:1319-1326.
22. Carpinteri A. Scaling laws and renormalization groups for strength and toughness of disordered materials. *International Journal of Solids and Structures*. 1994;31:291-302.
23. Carpinteri A, Spagnoli A. A fractal analysis of size effect on fatigue crack growth. *International Journal of Fatigue*. 2004;26:125-133.
24. Spagnoli A. Self-similarity and fractals in the Paris range of fatigue crack growth. *Mech Mater*. 2005;37:519-529.
25. Barenblatt GI, Botvina LR. Incomplete self-similarity of fatigue in the linear range of crack growth. *Fatigue Fract Eng Mater Struct*. 1980;3:193-202.
26. Barenblatt GI. *Scaling, Self-similarity and Intermediate Asymptotics*. UK: Cambridge University Press; 1996.
27. Barenblatt GI. *Scaling*. UK: Cambridge University Press; 2003.
28. Barenblatt GI. Scaling phenomena in fatigue and fracture. *International Journal of Fracture*. 2006;138:19-35.
29. Ritchie RO. Incomplete self-similarity and fatigue-crack growth. *International Journal of Fracture*. 2005;132:197-203.
30. Buckingham E. Model experiment and the form of empirical equations. *Transaction ASME*. 1915;37:263-296.
31. Ray S., Chandra Kishen JM. Fatigue crack propagation in plain concrete using incomplete self-similarity. In: Proceedings of 12th International Congress on Fracture, Ottawa; 2009: 1-10.
32. Carpinteri A, Paggi M. Self-similarity and crack growth instability in the correlation between the Paris' constants. *Engineering Fracture Mechanics*. 2007;74:1041-1053.
33. Carpinteri A, Paggi M. A unified interpretation of the power laws in fatigue and the analytical correlations between cyclic properties of engineering materials. *International Journal of Fatigue*. 2009;31:1524-1531.
34. Ciavarella M, Paggi M, Carpinteri A. One, no one, and one hundred thousand crack propagation laws: A generalized Barenblatt and Botvina dimensional analysis approach to fatigue crack growth. *J Mech Phys Solids*. 2008;56:3416-3432.
35. Carpinteri A. Fractal nature of material microstructure and size effects on apparent mechanical properties. *Mech Mater*. 1994;18:89-101. Internal Report, Laboratory of Fracture Mechanics, Politecnico di Torino, N. 1/92, 1992.
36. Wilson K. Renormalization group and critical phenomena. *Physical Review B*. 1971;4:3174-3205.
37. Carpinteri A, Ferro G. Size effects on tensile fracture properties: A unified explanation based on disorder and fractality of concrete microstructure. *Materials and Structures (RILEM)*. 1994;27:563-571.
38. Carpinteri A, Chiaia B. Power scaling laws and dimensional transitions in solid mechanics. *Chaos, Solitons & Fractals*. 1996;7:1343-1364.
39. Carpinteri A, Chiaia B, Invernizzi S. Three-dimensional fractal analysis of concrete fracture at the meso-level. *Theoretical and Applied Fracture Mechanics*. 1999;31:163-172.
40. Carpinteri A, Chiaia B. Multifractal nature of concrete fracture surfaces and size effects on nominal fracture energy. *Materials and Structures (RILEM)*. 1995;28:435-443.

41. Carpinteri A, Chiaia B. Multifractal scaling laws in the breaking behaviour of disordered materials. *Chaos, Solitons & Fractals*. 1997;8:135-150.
42. Williford R. Multifractal fracture. *Scripta Metallurgica et Materialia*. 1988;22:1749-1754.
43. Carpinteri A, Chiaia B. Size effects on concrete fracture energy: Dimensional transition from order to disorder. *Materials and Structures (RILEM)*. 1996;29:259-266.
44. Carpinteri A, Chiaia B. A fractal model for simulating the formation of microcracks in the fracture process zone and a theoretical explanation of the size effect of the fracture energy of concrete - Discussion on a paper by X.H. Ji, S.Y.N. Chan and N.Q. Feng. *Magazine of Concrete Research*. 1998;50:369-370.
45. Vasudeven AK, Sadananda K, Louat N. A review of crack closure, fatigue crack threshold and related phenomena. *Mater Sci Eng A*. 1994;188:1-22.
46. Mandelbrot BB, Passoja DE, Paullay AJ. Fractal character of fracture surfaces of metals. *Nature*. 1984;308:721-722.
47. Jones R, Chen F, Pitt S, Paggi M, Carpinteri A. From NASGRO to fractals: Representing crack growth in metals. *International Journal of Fatigue*. 2016;82:540-549.
48. Carpinteri A, Paggi M. A unified fractal approach for the interpretation of the anomalous scaling laws in fatigue and comparison with existing models. *International Journal of Fracture*. 2010;161:41-52.
49. Frost NE, Dugdale DS. The propagation of fatigue cracks in sheet specimens. *J Mech Phys Solids*. 1958;6:92-110.
50. Molent L, Spagnoli A, Carpinteri A, Jones R. Fractals and the lead crack airframe lifing framework. *Procedia Structural Integrity*. 2016;2:3081-3089.
51. Jones R, Molent L, Pitt S. Crack growth of physically small cracks. *International Journal of Fatigue*. 2007;29:1658-1667.
52. Jones R, Pitt S, Peng D. The generalized Frost-Dugdale approach to modelling fatigue crack growth. *Engineering Failure Analysis*. 2008;15:1130-1149.
53. Molent L, Spagnoli A, Carpinteri A, Jones R. Using the lead concept and fractal geometry for fatigue lifing of metallic structural components.*International Journal of Fatigue*. 2017;102:214-220.
54. Williford RE. Fractal fatigue. *Scripta Metallurgica et Materialia*. 1991;25:455-460.
55. Underwood E, Banerjee K. Fractal analysis of fracture surfaces. In: *ASM Handbook, Fractography*. Metals Park, Ohio: ASM International; 1987;12:211-215.
56. Handbook ASM. *8th Edition*. ASM International: Metals Park, Ohio; 1975;10.
57. Spagnoli A. Fractality in the threshold condition of fatigue crack growth: an interpretation of the Kitagawa diagram. *Chaos, Solitons & Fractals*. 2004;22:589-598.
58. Paggi M, Carpinteri A. Fractal and multifractal approaches for the analysis of crack-size dependent scaling laws in fatigue. *Chaos, Solitons & Fractals*. 2009;40:1136-1145.
59. Carpinteri A, Paggi M. Dimensional analysis and fractal modeling of fatigue crack growth. *Journal of ASTM International*. 2011;8:1-13.
60. El Haddad MH, Topper TH, Smith KN. Prediction of non propagating cracks. *Engineering Fracture Mechanics*. 1979;11:573-584.
61. Luo J, Bowen P. Small and long fatigue crack growth behaviour of a PM Ni-based superalloy, Udimet 720. *International Journal of Fatigue*. 2004;26:113-124.
62. Tanaka K, Nakai Y, Yamashita M. Fatigue growth threshold of small cracks. *International Journal of Fatigue*. 1981;17:519-533.
63. Usami S. Applications of threshold cyclic-plastic-zone-size criterion to some fatigue limit problems. In: Backlund J, Blom AF, Beevers CJ, eds. *Fatigue Threshold, Vol. I*. Warley: Engineering Materials Advisory Services; 1982:205-238.
64. Fleck NA, Kang KJ, Ashby MF. The cyclic properties of engineering materials. *Acta Metallurgica et Materialia*. 1994;42:365-381.

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