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ORIGINAL

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Static–kinematic duality in the shells of revolution: Historical aspects and present developments

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Abstract In this work, an historical overview concerning the theory of shell structures is presented. Early conjectures proposed by, among others, French, German, and Russian authors are discussed. Moreover, considering a recent approach in the field of structural analysis, based on the static–kinematic duality concept, static and kinematic matrix operator equations are formulated in the case of shells of revolution, emphasizing how these operators are one the adjoint of the other. In this way, any possible inaccuracy provided by the previous approaches can be overcome.

Keywords Static-kinematic duality · Shells of revolution · Theory of shell structures

1 Introduction

Modern shell theory has its roots in nineteenth-century structural mechanics. At first, the attention of the scientists focused on thin shells or membranes, rather than shells with flexural stiffness. Augustin-Louis Cauchy (1789–1857) is the forerunner for the study of thin cylindrical shells [1]. Siméon Denis Poisson (1781–1840), in his memory of 1829 [2], dedicated a chapter to the preliminary study of the shells of revolution, defining the equations for a thin shell stressed by forces tangential to its surface. The problem of a thin shell subjected to axially symmetric loading conditions was also examined by Lamé and Clapeyron [3].

Afterward, the attention of the mathematicians was attracted by the problem of a shell subjected to bending [4]. It is due to Aron [5] the first attempt to solve this problem in general terms, on the basis of Kirchhoff's kinematic hypotheses [6], expressing the geometry of the shell mean surface in parametric form, and deriving the equilibrium equations in analogy to the method used by Clebsch for plates [5,7].

Mathieu [8] applied the Poisson's equations to the case of a shell of revolution, assuming the deformations as negligible, and analyzing, in particular, the modes of vibration of a spherical shell. Further fundamental contributions to the dynamic problem of shells are due to Lord Rayleigh [9].

A general theory of shells is due to Love [10], based on the following hypotheses: shell thickness is small compared to its minimum radius of curvature; deformations and displacements are infinitesimal. Lamb [11] and Basset [12] solved Love's general equations for a cylindrical shell and demonstrated the possibility of a narrow boundary layer in which there is a rapid transition between flexural and membrane regimes. The stress analysis of cylindrical shells was settled later by Donnell [13], who advanced development of monocoque bodies for automobiles and planes.

The solution to the problem of the spherical shell of constant thickness and symmetric loading was provided by Reissner [14], who expressed the shell deformation in terms of differential equations of the second order,

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finding their solution by means of exponential functions. From Reissner's formulation, it follows that the flexural regime is mainly governed by forces acting on the shell's boundary [14].

In the following year, Meissner [15] noted that Reissner's equations could be generalized to include all the shells of revolution generated by curves with a variable radius of curvature. In a subsequent extension of his work [16], the author removed this latter limitation, defining a proper variation in thickness (Meissner condition). A general theory for shells of revolution applicable when the rise of the shell is small in comparison with the shell span, i.e., the case of shell roofs, was given by Marguerre [17].

2 The earlier German School: from Schwedler dome to Föppl's shell theory

Early approaches to shell structural analysis, such as the forward-looking *Lehrbuch der Statik* [18] by August Ferdinand Möbius (1790–1868), or Otto Mohr's (1835–1918) article on the composition of forces in space [19] attracted little interest from structural engineers. Even for thin shell structures, load transfer in three dimensions was initially not considered, but a plane frame approach was taken into account. One such structure, the gasometer of the Imperial Continental Gas Association (ICGA) located at Hellweg 8, Berlin, collapsed in 1860 while being erected. The engineer responsible for that project, Johann Wilhelm Schwedler (1823–1894), chief engineer for the Preußische Staatseisenbahnen (Royal Prussian Railways), improved the design of the dome structure that was rebuilt one year later, although he still used the conventional approach. In 1863, he designed another dome structure for the same client, covering the gasometer at Holzmarktstraße 28, Berlin, and became the first engineer to perform a three-dimensional structural analysis of a dome, which is known as "Schwedler dome" in the technical literature. Three years later, he described five further "Schwedler domes" in the journal Zeitschrift für Bauwesen, providing not only the theory behind them, but also a simplified structural calculation technique [4]. Schwedler considered a two-dimensional curved elastic continuum, an approach for which August Föppl (1854–1924), a Mohr's student in the Königliche Württembergische Baugewerkeschule in Stuttgart (Figs. 1, 2), developed a method to cover other shell-type structures [20,21], and which became obsolete only at the end of the 1960s when computers started to be used for analyzing spatial frameworks.

August Föppl's work was the fruit of his deductive research into the static of spatial frameworks that he started in 1881 [20], and remained substantially unsurpassed until the 1960s. Föppl, who succeeded to Johann Bauschinger at the Technischen Hochschule of München, proposed numerous forms of spatial structures such as shells, lattice domes, spatial bridge systems, and trussed systems braced in three dimensions. He also developed a new approach for analyzing familiar systems such as the Schwedler dome. His work is a

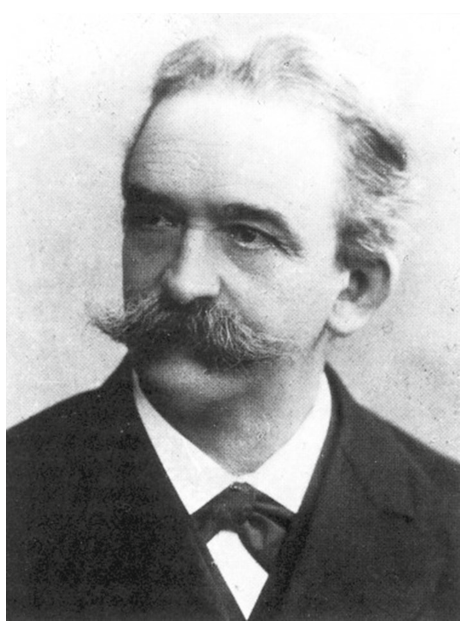

Fig. 1 Christian Otto Mohr (1835–1918)

Fig. 2 August Otto Föppl (1854–1924)

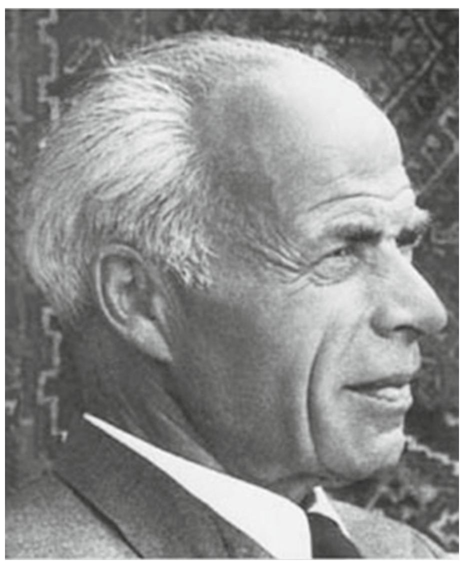

Fig. 3 Franz Anton Dischinger (1887–1953)

rare example of great heuristic potential and theoretical thinking, the horizon of which was not reached until recently [22,23].

Based on Föppl's considerations, Franz Anton Dischinger (1887–1953) [24], pupil of Friedrich Engesser (1848–1931), who was Mohr's student, as well as Föppl, at the Technischen Hochschule of Dresden, developed a method of calculation and construction of shells since 1923 [25,26]. He is considered one of the pioneers in the field of shell construction (Fig. 3). In 1942 Dischinger, after his collaboration with the German Bureau Dyckerhoff and Widmann AG, owner of the well-known DyWidAG patent, designed the monumental Berlin's Volkshalle in collaboration with Hitler's architect Albert Speer: a 250 m wide bombproof shell structure near the Reichstag, with a hosting capacity of more than 200 thousand people [27]. This project, never realized, was to be world's most impressive building in terms of its size and symbolic value.

3 The Russian reset: Timoshenko and Novozhilov

In the first half of the twentieth century, the theory of shell structures was rewritten and published in a more organic form by two Russian scientists: Stepan Prokof'evič Timoshenko (1878–1972), and Viktor Valentinovich Novozhilov (1892–1970).

Timoshenko (Fig. 4) completed his studies at the Institute of Engineers of Ways of Communication in St. Petersburg in 1901. He was professor at the Faculty of Civil Engineering at the Polytechnic Institute of Kiev from 1906 to 1911, being dismissed in connection with his dispute against the Tsar Nikolai II [22,23]. Then, according to Joffe [28], he joined the Gorki's group in the years leading to the revolution, and he probably was "the most left-wing of the Russian professors" [23,28,29]. In 1918, Timoshenko became professor at the Polytechnic Institute of Kiev, and one of the founders of the reorganized Ukraine Academy of Sciences.

In 1922, together with the creation of the Union of Soviet Socialist Republics, Timoshenko moved to the USA, where he was involved first at the Westinghouse Company and then as a professor at the Michigan University. From 1936 onward, he was professor at Stanford. He affected applied mechanics in the twentieth-century like no other, and had a decisive influence for the theory of shells [4,22].

In 1940, Timoshenko published for the first time a comprehensive treatise about shell theory, gathering all the previous scientific results [30].

Considering the equilibrium of a shell of revolution symmetrically loaded with respect to the axis of symmetry Z, and assuming the curvilinear coordinate s (Fig. 5), we have [30]:

$$\begin{bmatrix} \varepsilon_s \\ \varepsilon_\vartheta \\ \gamma_s \\ \chi_s \\ \chi_\vartheta \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}s} & \frac{1}{R_1} & 0 \\ +\frac{\sin\alpha}{r} & \frac{1}{R_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}s} \\ 0 & 0 & +\frac{\sin\alpha}{r} \end{bmatrix} \begin{bmatrix} u \\ w \\ \varphi_s \end{bmatrix} \tag{1}$$

Kinematic equations (1), and the corresponding differential matrix operator, are obtained by re-proposing in a matrix form the kinematic equations as they appear in Timoshenko [30].

Note that Timoshenko, also following Flügge [31], did not consider the deformation γ_s [31,32].

On the other hand, in 1947 Novozhilov, a professor of the Leningrad State University (Fig. 6) published one of the main contributions about the shell theory [33]. This comprehensive treatise, originally written in Russian, appeared in different subsequent English versions [34,35].

Fig. 4 Stepan Prokof'evič Timošenko (1878–1972)

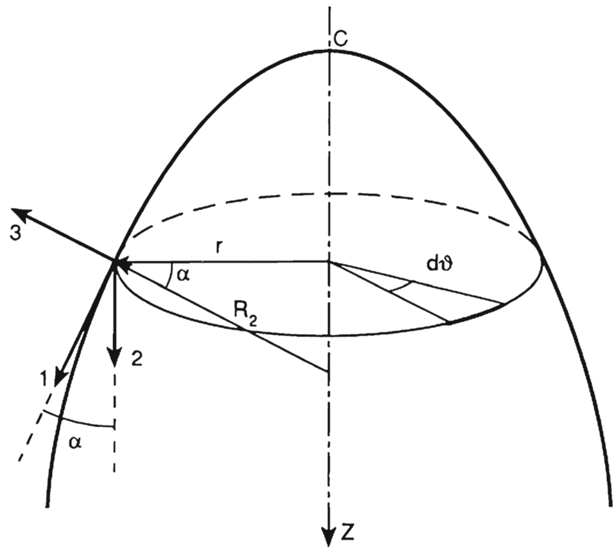

Fig. 5 Shells of revolution

According to Novozhilov [35], the kinematic equations, and the corresponding differential matrix operator, appear as in Eq. (1).

Note that, following Flügge [31], and Timoshenko [30], the shearing strain, γ_s, [36] was not taken into consideration by Novozhilov.

It is important to emphasize that the aforementioned treatises—the former by Timoshenko and the latter by Novozhilov—present to the reader a shortcoming in some terms in the kinematic matrix operator, in particular concerning the shearing strain. As can be seen by applying a more recent theoretical approach, the correct terms can be determined on the basis of the static–kinematic duality [37–39].

4 Static–kinematic duality in the shells of revolution

The static–kinematic duality leads to a simple and direct demonstration of the principle of virtual work for deformable bodies, and vice versa [37–40]. The two concepts imply each other [41]. Such an implication derives from the representation of the elastic problem in a symmetrical manner by combining the three fundamental relations: indefinite equations of equilibrium, kinematic equations as definition of deformation characteristics, and constitutive equations, in a single matrix operator equation where the unknown is represented by the displacement vector.

The definition of the static and kinematic matrix operator equations is possible for the shells of revolution, and it is verified that the kinematic matrix operator is the adjoint of the corresponding static matrix operator, and vice versa [39,42].

As regards the shells of revolution, the kinematic and static equations, with reference to Fig. 5, are as follows:

$$\begin{bmatrix} \varepsilon_s \\ \varepsilon_\vartheta \\ \gamma_s \\ \chi_s \\ \chi_\vartheta \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}s} & \frac{1}{R_1} & 0 \\ +\frac{\sin\alpha}{r} & \frac{1}{R_2} & 0 \\ -\frac{1}{R_1} & \frac{\mathrm{d}}{\mathrm{d}s} & +1 \\ 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}s} \\ 0 & 0 & +\frac{\sin\alpha}{r} \end{bmatrix} \begin{bmatrix} u \\ w \\ \varphi_s \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} \left(\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\sin\alpha}{r}\right) & -\frac{\sin\alpha}{r} & \frac{1}{R_1} & 0 & 0 \\ -\frac{1}{R_1} & -\frac{1}{R_2} & \left(\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\sin\alpha}{r}\right) & 0 & 0 \\ 0 & 0 & -1 & \left(\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\sin\alpha}{r}\right) & -\frac{\sin\alpha}{r} \end{bmatrix} \begin{bmatrix} N_s \\ N_\vartheta \\ T_s \\ M_s \\ M_\vartheta \end{bmatrix} + \begin{bmatrix} \mathcal{F}_s \\ \mathcal{F}_n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

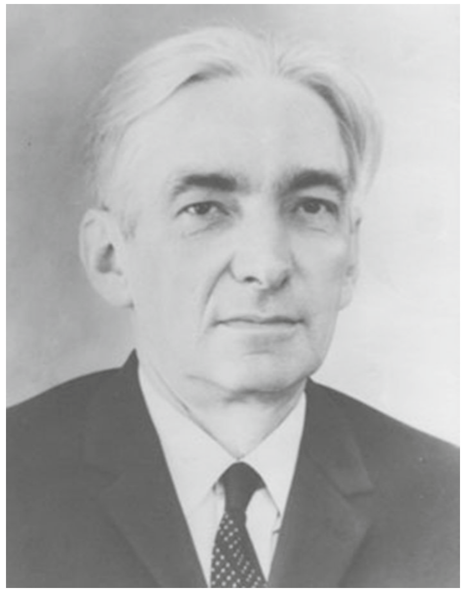

Fig. 6 Valentin Valentinovich Novozhilov (1892–1970)

Note that, for reasons of symmetry, the conditions of equilibrium to translation along the parallel and to rotation about the meridian are identically satisfied and thus do not appear in Eq. (3). We have three equations of equilibrium (respectively, with regard to translation along the meridian, to translation along the normal, and to rotation about the parallel) in the five static unknowns N_{s}, N_ϑ, T_{s}, M_{s}, M_ϑ.

The elastic problem for shells of revolution thus has two degrees of internal redundancy, whereas the more general problem of shells with double curvature appears to have three degrees of internal redundancy.

Equation (3) are verified by imposing the above three conditions of equilibrium on an infinitesimal shell element, bounded by two meridians located at an infinitesimal angular distance $r\mathrm{d}\vartheta$ and by two parallels located at an infinitesimal distance $\mathrm{d}s$.

The condition of equilibrium with regard to translation along the meridian yields the equation (Fig. 7a, c, e):

$$\mathrm{d}N_s r\mathrm{d}\vartheta + N_s \mathrm{d}r\mathrm{d}\vartheta - N_\vartheta \sin\alpha \mathrm{d}s\mathrm{d}\vartheta + T_s \frac{\mathrm{d}s}{R_1} r\mathrm{d}\vartheta + \mathcal{F}_s r\mathrm{d}s\mathrm{d}\vartheta = 0 \tag{4}$$

which, divided by $r\mathrm{d}s\mathrm{d}\vartheta$, coincides with the first of Eq. (3).

The condition of equilibrium with regard to translation along the normal n furnishes the equation (Fig. 7c–e):

$$-N_s \frac{\mathrm{d}s}{R_1} r\mathrm{d}\vartheta - N_\vartheta \mathrm{d}s\mathrm{d}\vartheta \cos\alpha + \mathrm{d}T_s r\mathrm{d}\vartheta + T_s \mathrm{d}r\mathrm{d}\vartheta + \mathcal{F}_n r\mathrm{d}s\mathrm{d}\vartheta = 0 \tag{5}$$

which, divided by $r\mathrm{d}s\mathrm{d}\vartheta$, coincides with the second of Eq. (3).

Finally, the condition of equilibrium with regard to rotation about the parallel furnishes the equation (Fig. 7b, c):

$$-T_s r\mathrm{d}\vartheta\mathrm{d}s + \mathrm{d}M_s r\mathrm{d}\vartheta + M_s \mathrm{d}r\mathrm{d}\vartheta - M_\vartheta \sin\alpha \mathrm{d}s\mathrm{d}\vartheta = 0 \tag{6}$$

which, divided by $r\mathrm{d}s\mathrm{d}\vartheta$, coincides with the third of Eq. (3).

Notice that in the indefinite equations of equilibrium (3), five terms $\sin\alpha/r$ are present. These contributions are due to the fact that the parallel curved sides of the shell element of Fig. 7a, b differ by the amount $\mathrm{d}r\mathrm{d}\vartheta$, as well as that the different action lines of the forces acting on the remaining two meridian sides are convergent and present a radial component.

While the static Eq. (3) can be obtained straightforwardly, the derivation of the kinematic Eq. (2), where the terms $\sin\alpha/r$ are partially absent (only the terms related to ε_ϑ, χ_ϑ, and to the meridian actions N_ϑ, M_ϑ are maintained), is comprehensively reported in [39], deriving the kinematic Eq. (2) from the principle of virtual work and the static Eq. (3).

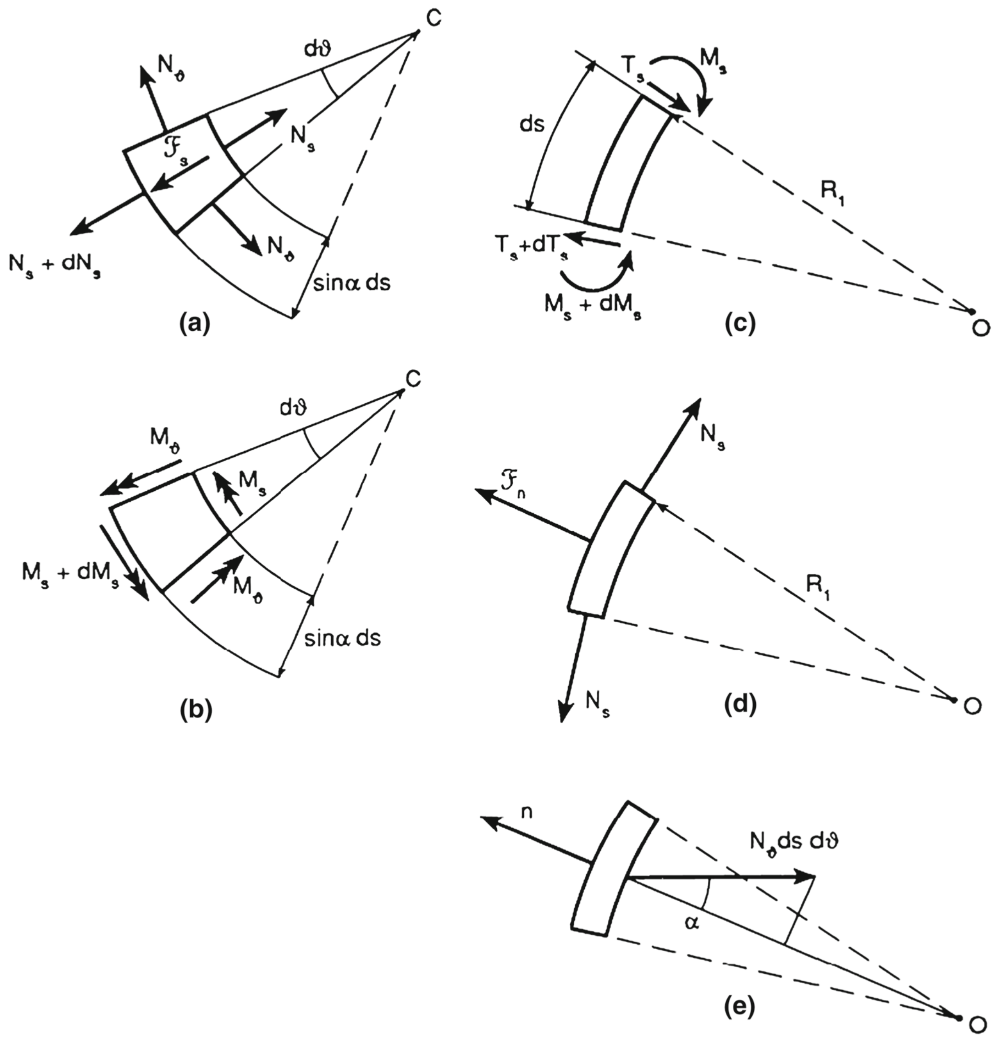

Fig. 7 Shells of revolution: infinitesimal element

Finally, it is worth noting that the terms of the kinematic matrix operator (see Eq. (2)) are correctly determined only referring to the static–kinematic duality. On the contrary, considering the foregoing classical approaches by Timoshenko, and by Novozhilov (see Eq. 1), some inaccuracy may arise in the analysis of the shells of revolution, in particular when dealing with the shearing strain.

5 Conclusions

This paper presents an historical overview of the theory of shell structures, starting from the early conjectures proposed by French, German, and English authors, up to the classical approaches proposed by the Russian School. Subsequently, taking into account a more recent approach in the field of structural analysis, based on the concept of static–kinematic duality, static and kinematic matrix operator equations are formulated in the case of the shells of revolution, emphasizing how these operators are one the adjoint of the other. It is worth noting that different terms of the kinematic matrix operator are correctly determined only by referring to the concept of static–kinematic duality. Conversely, considering the classical approaches, some inaccuracies did arise in the analysis of the shells of revolution.

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