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BRIEF NOTE

Cracks at rounded V-notch tips: an analytical expression for the stress intensity factor

A. Sapora · P. Cornetti · A. Carpinteri

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Abstract An analytical expression for the stress intensity factor related to a crack stemming from a blunted V-notch tip is put forward. The analysis is limited to mode I loading conditions, as long as the crack length is sufficiently small with respect to the notch depth. The proposed formula improves significantly the predictions of a recently introduced relationship, by considering a notch amplitude dependent parameter. Its values are estimated through a finite element analysis: different notch amplitudes, ranging from 0° to 180°, and different crack length to root radius ratios, ranging from 0 to 10, are taken into account. The evaluation of the apparent generalized fracture toughness according to equivalent linear elastic fracture mechanics concludes the paper.

Keywords Brittle fracture · Blunted notches · Mode I · SIF

1 Introduction

Several criteria have been recently introduced for the investigation of brittle fracture at rounded V-notched elements under mode I loading (Leguillon and Yosibash 2003; Gomez and Elices 2004; Taylor 2004; Pugno et al. 2004; Lazzarin and Berto 2005; Gomez et al. 2006; Picard et al. 2006; Ayatollahi and Torabi 2010; Carpinteri et al. 2011, 2012). These approaches, all involving a material length, differ from each other by the proposed failure condition: it can be a stress requirement (Taylor 2004; Pugno et al. 2004; Ayatollahi and Torabi 2010), an energy balance (Lazzarin and Berto 2005; Gomez et al. 2006) or it can be expressed by coupling both the considerations (Leguillon and Yosibash 2003; Picard et al. 2006; Carpinteri et al. 2011, 2012). In the latter case, the crack advance results a structural parameter, representing an additional unknown to be estimated.

While the stress condition can be easily implemented, since an accurate expression for the stress field along the notch bisector is available (Filippi et al. 2002), for what concerns energy based approaches the evaluation of the crack driving force still represents a drawback. The analysis reduces, according to Irwin's relationship, to the evaluation of the stress intensity factor (SIF) related to a crack at the notch tip, but no general analytical expressions are available in the Literature. The solution is thus obtained numerically by means of a finite element analysis (FEA) (Leguillon and Yosibash 2003; Picard et al. 2006).

Indeed, important results have been derived for U-notches (i.e. the rounded crack case, $\omega = 0°$) since the beginning of eighties (Schijive 1982; Lukas 1987; Xu et al. 1995). Lukas (1987), for instance, considered the generic geometry of an elliptical hole and proposed an approximating function for the SIF at the

A. Sapora (✉) · P. Cornetti · A. Carpinteri
Department of Structural, Geotechnical and Building Engineering, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Turin, Italy
e-mail: alberto.sapora@polito.it

notch root. This relationship has been widely applied to U-notches (which can be thought as limit cases to an elliptical hole, when the ratio between the length of the minor axis and that of the major axis tends to zero) by the scientific community (Gomez et al. 2006; Carpinteri et al. 2012), despite its range of validity is very restricted. In other words, the limitations of the hypothesis under which Lukas' formula was derived are often disregarded.

On the other hand, for what concerns generic notch amplitudes $\omega \neq 0°$, an important analytical contribution was presented in a previous work (Carpinteri et al. 2011). The expression was proposed to fulfil the asymptotic limits of very short and very long cracks. It was numerically verified for $\omega = 90°$, $120°$ and $150°$ (the three notch angles considered in the paper), showing the maximum deviation for $\omega = 90°$ (nearly 7 %). The error was expected to slightly increase for lower notch amplitudes.

In this work, an improved relationship is proposed, by introducing a novel parameter m, a function of the notch amplitude for $0° \leq \omega < 180°$. Its values are obtained via FEAs carried out by means of FRAN2D® code (Wawrzynek and Ingraffea 1991). The formula is then implemented by applying equivalent linear elastic fracture mechanics (LEFM) to predict the apparent critical generalized SIF (i.e. the apparent generalized fracture toughness, which is expected to be the governing failure parameter, (Carpinteri 1987; Gomez and Elices 2004; Lazzarin and Filippi 2006)) as a function of the root radius.

2 SIF function

Let us consider a crack of length c stemming from a blunted V-notch root (Fig. 1). The root radius is denoted by ρ. As far as the notch depth a is sufficiently large with respect to c, $a >> c$, the following SIF function was proposed (Carpinteri et al. 2011):

$$K_I(c) = \frac{\beta K_I^V c^{\lambda-1/2}}{\left[1 + \frac{q-1}{q}\left(\frac{\beta}{\psi}\right)^{\frac{1}{1-\lambda}} \frac{\rho}{c}\right]^{1-\lambda}} \tag{1}$$

The parameters β (Carpinteri et al. 2010), λ (Williams' eigenvalues), q and ψ, depend only on the notch

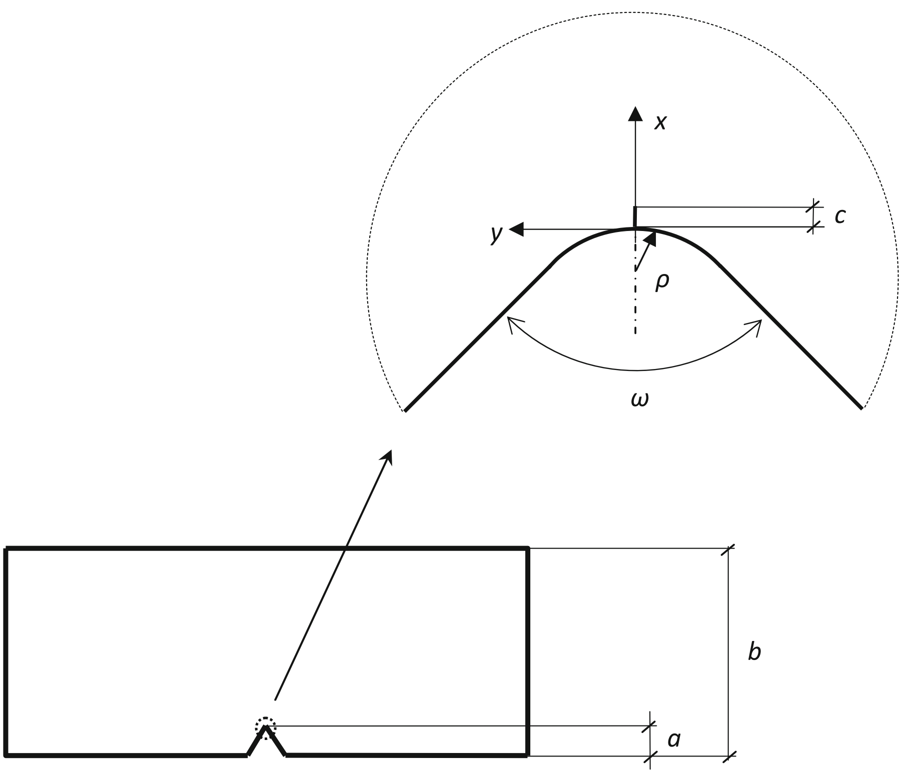

Fig. 1 Rounded V-notch with a crack stemming from the notch tip

Table 1 Notch amplitude ω dependent parameters

ω (°)	λ	β	$\eta_\theta(0)$	m
0	0.5000	1.000	1.000	1.820
30	0.5015	1.005	1.034	1.473
60	0.5122	1.017	0.9699	1.338
90	0.5445	1.059	0.8101	1.314
120	0.6157	1.161	0.5700	1.255
150	0.7520	1.394	0.2882	1.223
180	1.0000	1.985	–	–

amplitude ω. The former two have been reported in Table 1, the third reads

$$q = \frac{2\,\pi - \omega}{\pi}, \tag{2}$$

while the latter writes

$$\psi = 1.12\sqrt{\pi}\frac{[1 + \eta_\theta(0)]}{(2\,\pi)^{1-\lambda}}, \tag{3}$$

$\eta_\theta(0)$ varying as ω varies (Table 1, (Filippi et al. 2002)). Eventually, K_I^V in Eq. (1) represents the generalized SIF for a null radius $\rho = 0$.

Equation (1) can be easily written in dimensionless form as:

$$\bar{K}_I(\bar{c}) = \frac{\beta\,\bar{c}^{\lambda-1/2}}{\left[1 + \frac{q-1}{q}\left(\frac{\beta}{\psi}\right)^{\frac{1}{1-\lambda}}\frac{1}{\bar{c}}\right]^{1-\lambda}}, \tag{4}$$

where $\bar{K}_I(\bar{c}) = \frac{K_I(\bar{c})}{K_I^V\,\rho^{\lambda-1/2}}$ and $\bar{c} = c/\rho$. As already stated, Eq. (4) was found to provide good results for $\omega = 90°$, $120°$ and $150°$: its predictions were compared with numerical results (Carpinteri et al. 2011), showing an error ranging from 3 % ($\omega = 150°$) to nearly 7 % ($\omega = 90°$).

In order to improve the accuracy of Eq. (4), investigating also lower notch amplitudes, the following generalization is now proposed, by introducing an additional parameter $m = m(\omega)$:

$$\bar{K}_I(\bar{c}) = \frac{\beta\,\bar{c}^{\lambda-1/2}}{\left\{1 + \left[\frac{q-1}{q}\left(\frac{\beta}{\psi}\right)^{\frac{1}{1-\lambda}}\frac{1}{\bar{c}}\right]^m\right\}^{\frac{1-\lambda}{m}}}. \tag{5}$$

Expression (5) reverts to (4) for $m = 1$ and it fulfils the asymptotic limits independently of m. In fact, for a very long crack, $\bar{c} >> 1$, but still small with respect to the notch depth a, Eq. (5) yields:

$$\bar{K}_I(\bar{c}) = \beta\,\bar{c}^{\lambda-1/2}, \tag{6}$$

which provides the SIF related to a crack at the V-notch tip (Hasebe and Iida 1978; Philipps et al. 2008), $K_I(c) = \beta\,K_I^V c^{\lambda-1/2}$. In other words, when $\bar{c} >> 1$, the root radius effect becomes negligible. On the other hand, for a very short crack, $\bar{c} << 1$, Eq. (5) leads to:

$$\bar{K}_I(\bar{c}) = \left(\frac{q}{q-1}\right)^{1-\lambda}\psi\,\sqrt{\bar{c}}, \tag{7}$$

i.e. it reverts to the expression for an edge crack subjected to the local peak stress, $K_I(c) = 1.122\,\sigma_{max}\sqrt{\pi\,c}$, where $\sigma_{max} = K_I^V[1 + \eta_\theta(0)]/[2\,\pi\,\rho(q-1)/q]^{1-\lambda}$ is the maximum stress attained at the notch tip (Filippi et al. 2002).

The values for the parameter m to be inserted into Eq. (5) will be evaluated according to the numerical procedure described in the following section.

3 FEA validation

In order to estimate the parameters m in Eq. (5), a FEA is carried out through FRAN2D® code (Wawrzynek and Ingraffea 1991). For each notch amplitude (ω ranging from 0° to 150°, with a step of 30°), an element with the following geometric ratios is considered: $\rho/a = a/b = 0.01$, being b the characteristic dimension of the sample (Fig. 1). Under such assumptions, the boundary effects can be reasonably disregarded. The mesh is generated by Q8 quadrangular elements (Fig. 2a refers to the case $\omega = 90°$), and it is significantly refined near the notch root (Fig. 2b). A unite tensile load is applied, and the stress field ahead of the notch tip is verified to match the analytical expression proposed in the Literature (Filippi et al. 2002). A crack along the bisector is then created: the mesh next to the root is automatically regenerated by the software through T6 triangular elements (Fig. 2c). The SIFs are evaluated through a displacement correlation technique. Different values for $\bar{c}$, from 0.05 to 10, have been tested: indeed, since for $\bar{c} \simeq 2$ results nearly approach the asymptotic limit of long cracks (Eq. (6)), only the range $0 \leq \bar{c} \leq 2$ will be considered in the present analysis. The study is presented in Figs. 3, 4, 5, 6, 7, 8 and 9: the asymptotic limits provided by Eqs. (6) and (7) are denoted by a dashed line and a dotted line, respectively. Numerical data are represented by circles, while results according to Eq. (4) are described by the continuous thin line. Eventually, predictions related to the proposed formula (5) refer to

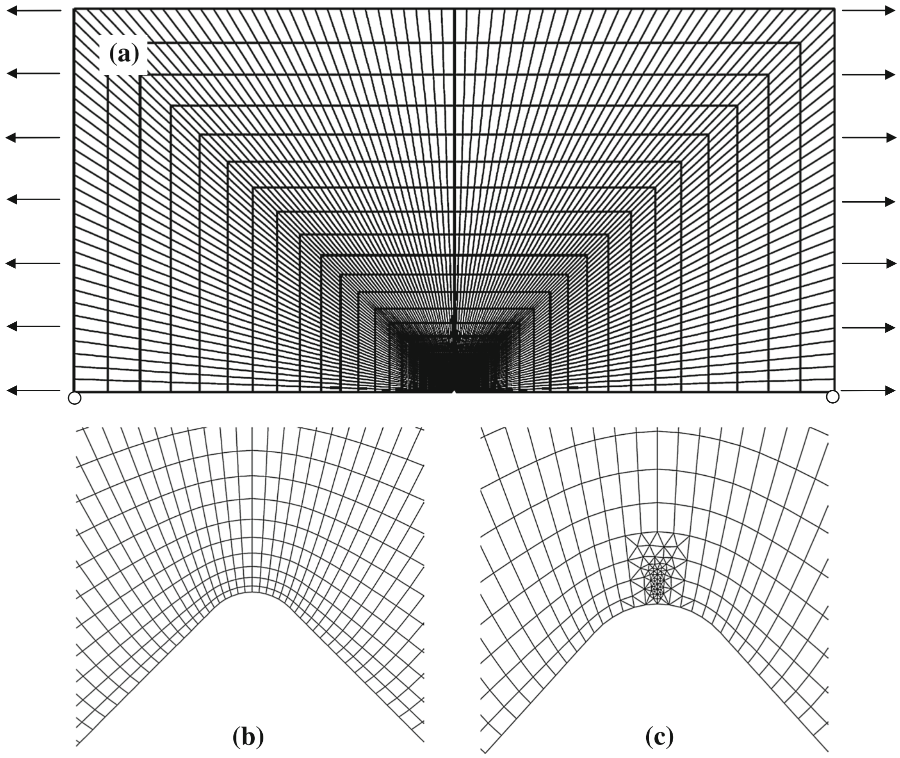

Fig. 2 Mesh for the 90°-notch amplitude sample (**a**). Mesh refinement at the notch root (**b**). Mesh reconstruction after the crack introduction (**c**)

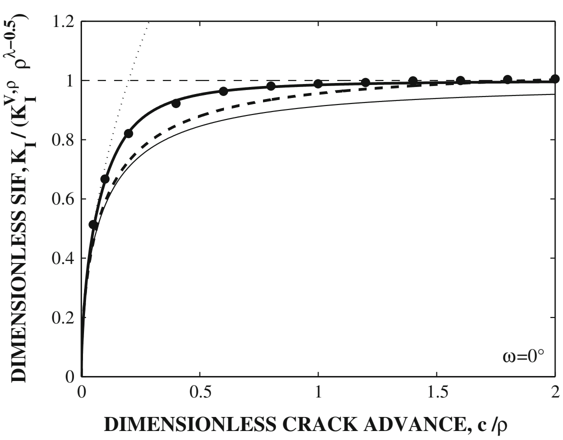

Fig. 3 Dimensionless SIF versus dimensionless crack length for ω = 0°: asymptotic limit for large cracks (Eq. (6), *dashed line*); asymptotic limit for small cracks (Eq. (7), *dotted line*); predictions according to Eq. (4) (*continuous thin line*); predictions according to Lukas' formula (Eq. (9), *thick dashed line*); predictions according to the proposed expression (Eq. (5), *continuous thick line*); numerical data (*circles*)

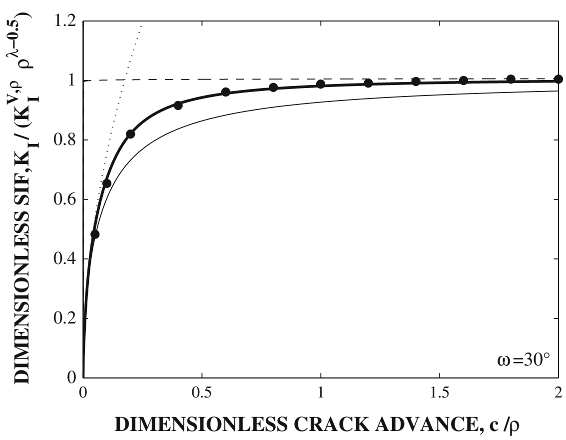

Fig. 4 Dimensionless SIF versus dimensionless crack length for ω = 30°: asymptotic limit for large cracks (Eq. (6), *dashed line*); asymptotic limit for small cracks (Eq. (7), *dotted line*); predictions according to Eq. (4) (*continuous thin line*); predictions according to the proposed expression (Eq. (5), *continuous thick line*); numerical data (*circles*)

the continuous thick line: they are obtained by means of the parameters m reported in Table 1, estimated using an iterative least squares method to improve the fitting with FEA.

Let us start by discussing the results related to the crack case ($\omega = 0°$ and $K_I^V = K_I^U$, K_I^U being the SIF of the corresponding crack, (Glinka 1985)), according to which Eq. (5) provides

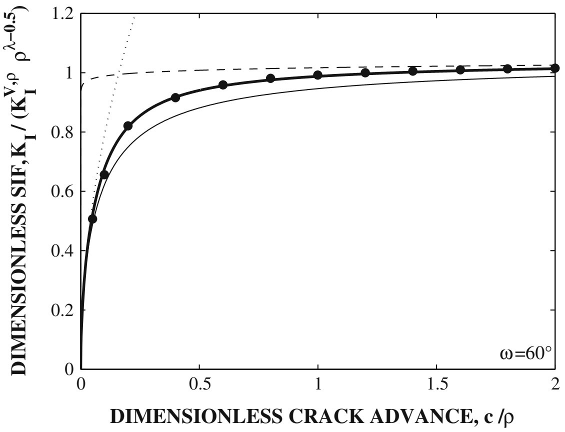

Fig. 5 Dimensionless SIF versus dimensionless crack length for ω = 60°: asymptotic limit for large cracks (Eq. (6), *dashed line*); asymptotic limit for small cracks (Eq. (7), *dotted line*); predictions according to Eq. (4) (*continuous thin line*); predictions according to the proposed expression (Eq. (5), *continuous thick line*); numerical data (*circles*)

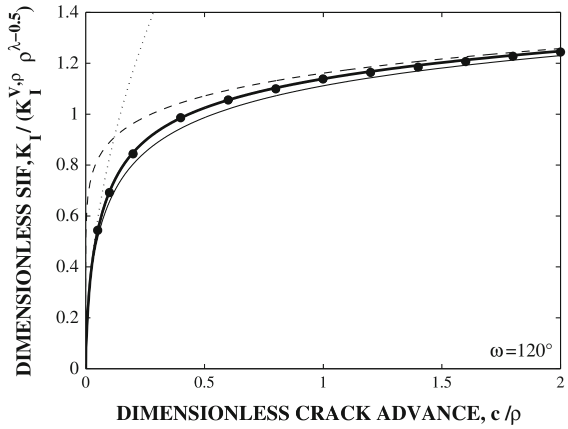

Fig. 7 Dimensionless SIF versus dimensionless crack length for ω = 120°: asymptotic limit for large cracks (Eq. (6), *dashed line*); asymptotic limit for small cracks (Eq. (7), *dotted line*); predictions according to Eq. (4) (*continuous thin line*); predictions according to the proposed expression (Eq. (5), *continuous thick line*); numerical data (*circles*)

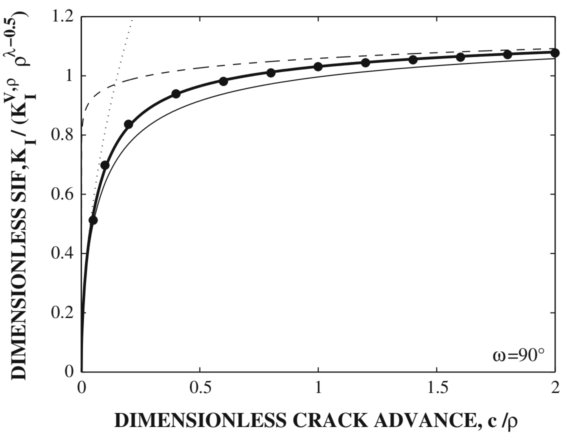

Fig. 6 Dimensionless SIF versus dimensionless crack length for ω = 90°: asymptotic limit for large cracks (Eq. (6), *dashed line*); asymptotic limit for small cracks (Eq. (7), *dotted line*); predictions according to Eq. (4) (*continuous thin line*); predictions according to the proposed expression (Eq. (5), *continuous thick line*); numerical data (*circles*)

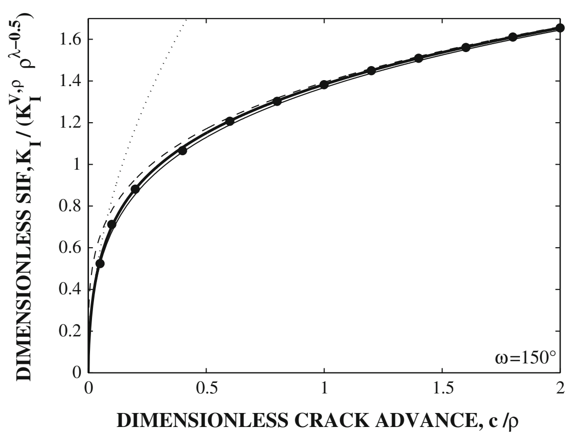

Fig. 8 Dimensionless SIF versus dimensionless crack length for ω = 150°: asymptotic limit for large cracks (Eq. (6), *dashed line*); asymptotic limit for small cracks (Eq. (7), *dotted line*); predictions according to Eq. (4) (*continuous thin line*); predictions according to the proposed expression (Eq. (5), *continuous thick line*); numerical data (*circles*)

$$\bar{K}_I(\bar{c}) = 2.243\sqrt{\frac{\bar{c}}{[1+(5.031\bar{c})^m]^{\frac{1}{m}}}}, \tag{8}$$

while the asymptotic limits (6) and (7) revert to $\bar{K}_I(\bar{c}) = 1$ and $\bar{K}_I(\bar{c}) = 2.243\sqrt{\bar{c}}$, respectively. The perfect matching between theoretical predictions related to $m = 1.82$ and numerical data is evident from Fig. 3. Notice that also the estimations according to Lukas' formula (providing errors within 5 % as long as $c/\rho < 0.2$, Lukas 1987),

$$\bar{K}_I(\bar{c}) = 2.243\sqrt{\frac{\bar{c}}{1+4.5\bar{c}}}, \tag{9}$$

have been drawn (thick dashed line). In Eq. (9), a factor 4.5 instead of 5.031 was considered to improve the fitting for short cracks. On the other hand, for long cracks, Eq. (9) provides $\bar{K}_I(\bar{c}) \simeq 1.056$ with an overestimation of more than 5 %. By looking at Fig. 3, the error committed implementing Lukas' relationship outside its range of validity is evident.

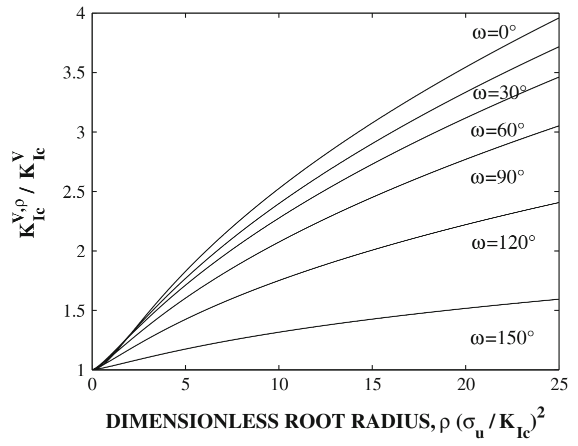

Fig. 9 Dimensionless apparent generalized fracture toughness versus dimensionless notch root radius, for different notch amplitudes ω. Results refer to equivalent LEFM

Increasing the notch amplitude (Figs. 4, 5, 6, 7 and 8), the value for m to improve the precision of Eq. (5) decreases (Table 1) and, consequently, the discrepancy with estimations according to Eq. (4) decreases as well. Nevertheless, the improvement of Eq. (5) keeps to be significant, reducing the maximum deviation below 1–2 %.

Eventually, for the flat edge case ($\omega = 180°$), Eq. (5) provides coherently

$$\bar{K}_I(\bar{c}) = 1.122\sqrt{\pi\,\bar{c}}, \tag{10}$$

which leads to $K_I(c) = 1.122\sigma\sqrt{\pi\,c}(K_I^V = \sigma$ for $\omega = 180°)$, i.e. the SIF does not depend on the root radius ρ. In this case, no numerical simulations are obviously mandatory.

4 Equivalent LEFM: apparent generalized fracture toughness

In this section, the generalization of the well known Griffith's criterion (i.e. equivalent LEFM) is applied to estimate the apparent generalized fracture toughness for brittle rounded V-notched structures. According to the approach, failure takes place if the crack driving force G equals the fracture energy G_c at a distance $\Delta = \frac{1}{\pi}\left(\frac{K_{Ic}}{\sigma_u}\right)^2$ from the notch tip. Thus, the crack advance Δ results in a material property, function only of the fracture toughness K_{Ic} and the tensile strength σ_u. By exploiting Irwin's relationship under plain strain conditions, the failure condition reads:

$$K_I(\Delta) = K_{Ic}. \tag{11}$$

Substituting the dimensional form of Eq. (5) into Eq. (11), simple analytical manipulations yield:

$$\bar{K}_{Ic}^{V,\rho} = \left\{1 + \left[\frac{q-1}{q}\left(\frac{\beta}{\psi}\right)^{\frac{1}{1-\lambda}}\pi\,\bar{\rho}\right]^m\right\}^{\frac{1-\lambda}{m}}, \tag{12}$$

where $\bar{K}_{Ic}^{V,\rho}$ is the apparent generalized fracture toughness normalized with respect to the generalized fracture toughness $K_{Ic}^V = \frac{K_{Ic}}{\beta\,\Delta^{\lambda-1/2}}$, while $\bar{\rho} = \rho\left(\frac{\sigma_u}{K_{Ic}}\right)^2$ is the dimensionless notch root radius. Predictions according to Eq. (12) are reported in Fig. 9, showing that the apparent generalized fracture toughness increases as the root radius ρ increases (for a fixed ω) and as the notch amplitude ω decreases (for a fixed ρ).

5 Conclusions

An improved formula for the evaluation of the SIF function related to a crack stemming from a blunt V-notch root was proposed. The accuracy was validated by numerical simulations, carried out for different notch amplitudes and different crack lengths. The expression was then implemented for the theoretical investigation of failure in rounded V-notched elements, in the spirit of equivalent LEFM: a simple analytical expression for the apparent generalized fracture toughness was derived. Indeed, the present formula can be easily applied also to more sophisticated fracture criteria, where the crack driving force has to be integrated over the crack advance for satisfying failure conditions (Leguillon 2002; Pugno and Ruoff 2004; Carpinteri et al. 2008).

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