ex: discuss the solutions of the following linear system as $\alpha \in \mathbb{R}$, and, when possible, find the solutions:

$$\begin{cases} x+y+\alpha z = 2 \\ x+y+3z = \alpha - 1 \\ 2x+\alpha y - z = 1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & \alpha & 2 \\ \boxed{1} & 1 & 3 & \alpha-1 \\ 2 & \alpha & -1 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & \alpha & 2 \\ 0 & 0 & 3-\alpha & \alpha-3 \\ 2 & \alpha & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & \alpha & 2 \\ 0 & 0 & 3-\alpha & \alpha-3 \\ 0 & \alpha-2 & -1-2\alpha & -3 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & \alpha & 2 \\ 0 & \alpha-2 & -1-2\alpha & -3 \\ 0 & 0 & 3-\alpha & \alpha-3 \end{array}\right)$$

If $\alpha = 3$ $\rightsquigarrow$ $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -7 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$rkA = 2$

$rk(A|B) = 2$

There are $3-2=1$ degree of freedom: ∞^1 solutions

$$\begin{cases} x+y+3z = 2 \\ y-7z = -3 \\ 0 = 0 \end{cases}$$

$$\begin{cases} x+y+3z=2 \Longrightarrow \boxed{x = 2-y-3z} = 5-10t \\ y-7z=-3 \Longrightarrow y = -3+7z = -3+7t \\ z = t \end{cases}$$

$$x = 2-(-3+7t)-3t = 5-10t$$

$$\begin{cases} x = 5-10t \\ y = -3+7t \\ z = t \end{cases} \quad t \in \mathbb{R}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & \alpha & 2 \\ 0 & \alpha-2 & -1-2\alpha & -3 \\ 0 & 0 & 3-\alpha & \alpha-3 \end{array}\right)$$

If $\boxed{\alpha \neq 3}$ and $\alpha \neq 2$

$\Longrightarrow rk(A)=3 \Longrightarrow rk(A|B)=3$

there are $\infty^{3-3} = \infty^{0} = 1$ solution

For example: if $\alpha = 0$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -2 & -1 & -3 \\ 0 & 0 & 3 & -3 \end{array}\right) \rightsquigarrow \begin{cases} x+y=2 \\ -2y-z=-3 \\ 3z=-3 \end{cases}$$

$$\begin{cases} x = 2-y = 0 \\ -2y = -3+z = -4 \Longrightarrow y = 2 \\ z = -1 \end{cases} \qquad \begin{cases} x=0 \\ y=2 \\ z=-1 \end{cases}$$

$\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ is the unique solution

If $\boxed{\alpha = 2}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 0 & -5 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + \frac{1}{5}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 0 & -5 & -3 \\ 0 & 0 & 0 & -8/5 \end{array}\right) \longrightarrow 0 = -8/5$$

$rk(A) = 2$

$rk(A|B) = 3$

$\Longrightarrow$ NO SOLUTIONS