HW : $V = \mathbb{R}[x]_2$ $q_1(x) = 1 - x^2$
$q_2(x) = 2 + x$

Prove that q_1 and q_2 are linearly indep,
but they do not generate.
If we add: $q_3(x) = -1 + x + x^2$
then prove that (q_1, q_2, q_3) is a basis of $\mathbb{R}[x]_2$.

• q_1 and q_2 are linearly indep:
if $\alpha_1 q_1 + \alpha_2 q_2 = 0$ then

$$\alpha_1 (1 - x^2) + \alpha_2 (2 + x) = 0$$

$$\alpha_1 + 2\alpha_2 + \alpha_2 x - \alpha_1 x^2 = 0$$

$$\begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ \alpha_2 = 0 \\ -\alpha_1 = 0 \end{cases} \rightsquigarrow \alpha_1 = \alpha_2 = 0 \quad \checkmark$$

On the other hand, it is not true that they generate all polys of deg ≤ 3:

$$p(x) = a_0 + a_1 x + a_2 x^2 \overset{?}{\Longrightarrow} \exists\, \alpha_1, \alpha_2 \text{ s.t. } \alpha_1 q_1 + \alpha_2 q_2 = p$$

$$\alpha_1 (1 - x^2) + \alpha_2 (2 + x) = a_0 + a_1 x + a_2 x^2$$

$$\alpha_1 + 2\alpha_2 + \alpha_2 x - \alpha_1 x^2 = a_0 + a_1 x + a_2 x^2$$

$$\begin{cases} \alpha_1 + 2\alpha_2 = a_0 \\ \alpha_2 = a_1 \\ -\alpha_1 = a_2 \end{cases} \Longrightarrow \text{if } a_0 \neq 2a_1 - a_2 \text{ the system has no solutions}$$

• q_1, q_2, q_3 are lin. indep. <u>and</u> they generate: since they are 3 vectors in a space of dim. 3, it would be enough to check one of the conditions, but it is easy to see that they are equivalent:

q_1, q_2, q_3 lin. indep

$\iff \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 = 0 \implies \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\iff \alpha_1 (1 - x^2) + \alpha_2 (2 + x) + \alpha_3 (-1 + x + x^2) = 0$

$(\alpha_1 + 2\alpha_2 - \alpha_3) + (\alpha_2 + \alpha_3) x + (\alpha_3 - \alpha_1) x^2 = 0$

$\implies \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\iff$ the only solution to the lin. system

$$\begin{cases} \alpha_1 + 2\alpha_2 - \alpha_3 = 0 \\ \alpha_2 + \alpha_3 = 0 \\ \alpha_3 - \alpha_1 = 0 \end{cases}$$

is $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\iff \operatorname{rk} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = 3$$

$\iff$ the lin. system

$$\begin{cases} \alpha_1 + 2\alpha_2 - \alpha_3 = a_0 \\ \alpha_2 + \alpha_3 = a_1 \\ \alpha_3 - \alpha_1 = a_2 \end{cases}$$

has solutions for any $a_0, a_1, a_2 \in \mathbb{R}$.

$\iff$ the equation $\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 = p$ has solutions for any poly $p(x) = a_0 + a_1 x + a_2 x^2 \in \mathbb{R}[x]_2$

$\iff q_1, q_2, q_3$ generate $\mathbb{R}[x]_2$

Since indeed $\operatorname{rk} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = 3$, we have a basis.

HW: prove Steinitz lemma:
Let V be a vector space over a field $\mathbb{K}$.
If $v_1 \dots v_n \in V$ are generators, and $w_1 \dots w_m \in V$ are linearly independent $\Rightarrow m \leq n$.

proof: if $m > n$, one can see that there are linear combinations

$$\lambda_1 w_1 + \dots + \lambda_m w_m = 0_V$$

where not all $\lambda_i = 0$, that is, if $m > n$ then the $w_1 \dots w_m$ must be lin. dep.

Since the v_i's are generators, any w_j can be written as a linear combination of them: $\forall j$, $\exists\ \alpha_{j1}, \dots, \alpha_{jn} \in \mathbb{K}$ such that

$$w_j = \alpha_{j1} v_1 + \alpha_{j2} v_2 + \dots + \alpha_{jn} v_n.$$

So
$$\begin{aligned}
&\lambda_1 w_1 + \lambda_2 w_2 + \dots + \lambda_m w_m \\
&= \lambda_1(\alpha_{11} v_1 + \dots + \alpha_{1n} v_n) + \lambda_2(\alpha_{21} v_1 + \dots + \alpha_{2n} v_n) \\
&\qquad + \dots + \lambda_m(\alpha_{m1} v_1 + \dots + \alpha_{mn} v_n) \\
&= (\lambda_1 \alpha_{11} + \lambda_2 \alpha_{21} + \dots + \lambda_m \alpha_{m1}) v_1 \\
&\quad + (\lambda_1 \alpha_{12} + \dots + \lambda_m \alpha_{m2}) v_2 \\
&\quad + \dots + (\lambda_1 \alpha_{1n} + \dots + \lambda_m \alpha_{mn}) v_n = 0
\end{aligned}$$

Now notice that the system

$$\begin{cases}
\lambda_1 \alpha_{11} + \lambda_2 \alpha_{21} + \dots + \lambda_m \alpha_{m1} = 0 \\
\lambda_1 \alpha_{12} + \dots + \lambda_m \alpha_{m2} = 0 \\
\vdots \\
\lambda_1 \alpha_{1n} + \dots + \lambda_m \alpha_{mn} = 0
\end{cases}$$

is a homogeneous system of n equations in m variables, so if $m > n$ has infinitely many solutions, meaning precisely that the $w_1 \dots w_m$ are linearly dependent.