Linear algebra and geometry, a.y. 2024-2025

LAG's exam simulation, written test: 8 multiple choice quizzes and 1 exercise

DIRECTIONS: this is a simulation of the written test for Linear algebra and geometry. **Since it is a simulation, it should not be considered as representative of the program of the course, neither it should be an indication of the subjects that could appear in the examination.** If you have any questions send me an email and ask.

Let me also remind you that:

- You are admitted to the written test only after passing the computer based test on the numerical analysis part, that is after obtaining at least 4.25 (out of 10) points in the first test.
- The written test concerns topics from the theoretical/geometry part, and it is worth a total of 23 points: 2 points for each quiz + a maximum of 7 points for the exercise. There is no penalty for wrong quizzes. You need to get at least 7 points in the quizzes part and at least 2 points in the exercise to pass the written test.
- The final grade is the sum of the points obtained in the two tests. The exam is passed if the grade is at least 18/30; the "lode" is given to whomever obtains 33/30.
- You can always check the exam rules on the teaching portal by clicking here, or reading my exam handbook.

QUIZZES

1. If the linear system $AX = B$ with n equations in n unknowns has at least two solutions, then

   (a) $\det(A) = \det(A^2)$.
   (b) $\mathrm{rk}(A) = n - 1$.
   (c) $\mathrm{rk}(A) = 0$.
   (d) $\det(A) \neq 0$.

2. Let $A \in \mathbb{R}^{4,4}$ be a matrix whose characteristic polynomial is $p_A(t) = t(t-2)(t^2+9)$.

   Find the true statement.

   (a) The matrix A has rank 4.
   (b) There exists an invertible matrix $P \in \mathbb{R}^{4,4}$ such that $P^{-1}AP$ is diagonal.
   (c) The matrix A might be symmetric.
   (d) There exists an invertible matrix $P \in \mathbb{C}^{4,4}$ such that $P^{-1}AP$ is diagonal.

3. In the space of applied vectors $V_3(O)$, consider the elements $\vec{u} = \vec{\imath} + 2\vec{\jmath} - \vec{k}$ and $\vec{v} = 2\vec{\imath} - \vec{\jmath} + \vec{k}$.

   Find the true statement.

   (a) $\vec{u} \times \vec{v}$ is orthogonal to the vector having coordinates $(-1, 3, 5)$.
   (b) $\vec{u} \times \vec{v}$ is parallel to the vector having coordinates $(2, -1, 1)$.
   (c) $\vec{u} \times \vec{v}$ is orthogonal to the vector having coordinates $(2, 4, -2)$.
   (d) $\vec{u} \times \vec{v}$ forms an obtuse angle with the vector having coordinates $(0, 1, -1)$.

4. Find the negative definite quadratic form.

   (a) $x^2 + y^2 - 100xy$.
   (b) $y^2 - x^2 - 100xy$.
   (c) $2xy - 3x^2 - 2y^2$.
   (d) $-2x^2 - y^2 - 6xy$.

5. Let (e_1, e_2, e_3) be the canonical basis of $\mathbb{R}^3$, and define the vectors

$$v_1 = e_1 + e_2 + e_3, \qquad v_2 = -3e_2, \qquad v_3 = 2e_1 + 5e_2 + 2e_3, \qquad v_4 = e_3 - e_1.$$

Find the true statement.

(a) The set (v_1, v_2, v_3) is a basis of $\mathbb{R}^3$.
(b) The set (v_1, v_2, v_3, v_4) generates $\mathbb{R}^3$.
(c) The set (v_1, v_3) generates $\mathbb{R}^3$.
(d) The set (v_1, v_2, v_3) is linearly independent in $\mathbb{R}^3$.

6. Given

$$B = \begin{pmatrix} 3 & 7 \\ 1 & -3 \end{pmatrix},$$

which of the following statements is true?

(a) There exists $X \in \mathbb{R}^{2,2}$ nonzero such that $BX = 3X$.
(b) $B^2 = B + 16I_2$.
(c) None of the other statements is true.
(d) If $A \in \mathbb{R}^{2,2}$ and the equation $AX = B$ has solution(s), then $\det(A) \neq 0$.

7. In the space S_3 with a fixed coordinate system, consider the plane $\pi : x + y - z = 0$.
Find the true statement.

(a) The plane with equation $x - y - z = 1$ is not parallel to π.
(b) The point $(1, 1, 2)$ does not belong to π.
(c) The point $(1, -1, 1)$ belongs to π.
(d) The line with equations $x + y - z = 3y = 0$ is not contained in π.

8. Let $f : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear map.
Find the true statement.

(a) The map f is injective.
(b) If $\dim(\mathrm{Ker}(f)) = 2$ then f is surjective.
(c) The map f is invertible.
(d) If $\dim(\mathrm{Ker}(f)) = 3$ then f is surjective.

EXERCISE

Let $\mathbb{R}[x]_2$ be the vector space of polynomials in one variable with real coefficients and degree at most 2. Consider then the matrix

$$A = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and the ordered set $\mathcal{B} = (1 + x^2, 2x^2 - 1, x)$ of elements in $\mathbb{R}[x]_2$.

(a) Explain why $\mathcal{B}$ is a basis of $\mathbb{R}[x]_2$.

(b) Let $f\colon \mathbb{R}[x]_2 \to \mathbb{R}[x]_2$ be the endomorphism whose matrix relative to the basis $\mathcal{B}$ both in the domain and codomain is $M_{\mathcal{B}}^{\mathcal{B}}(f) = A$: is f injective? Why or why not?

(c) Check that $1 - 2x + 4x^2 \in \mathrm{Ker}(f)$ and find a basis of $\mathrm{Ker}(f)$.

(d) Find all eigenvalues of f and compute their algebraic and geometric multiplicities.

(e) Is $M_{\mathcal{B}}^{\mathcal{B}}(f)$ diagonalizable? Why or why not?

(f) Find, if it exists, an element $p \in \mathbb{R}[x]_2$ such that $f(p) = x$.