Linear algebra and geometry, a.y. 2024-2025

LAG's exam simulation, written test: 8 multiple choice quizzes and 1 exercise

DIRECTIONS: this is a simulation of the written test for Linear algebra and geometry. **Since it is a simulation, it should not be considered as representative of the program of the course, neither it should be an indication of the subjects that could appear in the examination.** If you have any questions send me an email and ask.

Let me also remind you that:

- You are admitted to the written test only after passing the computer based test on the numerical analysis part, that is after obtaining at least 4.25 (out of 10) points in the first test.
- The written test concerns topics from the theoretical/geometry part, and it is worth a total of 23 points: 2 points for each quiz + a maximum of 7 points for the exercise. There is no penalty for wrong quizzes. You need to get at least 7 points in the quizzes part and at least 2 points in the exercise to pass the written test.
- The final grade is the sum of the points obtained in the two tests. The exam is passed if the grade is at least 18/30; the "lode" is given to whomever obtains 33/30.
- You can always check the exam rules on the teaching portal by clicking here, or reading my exam handbook.

QUIZZES

1. Consider the points

$$A = (1, 1, 1), \qquad B = (3, 3, 7), \qquad C = (0, 3, -2), \qquad D = (2, 2, 4).$$

   Which of the following statements is true?

   (a) The points$A$, B, C, D are not coplanar.
   (b) The points A, B, D are not collinear.
   (c) There exist infinitely many planes containing A, B, D.
   (d) None of the other statements is true.

2. Consider the plane

$$\alpha : \ x + 2y + 2z = 0$$

   and the line

$$r : \begin{cases} x = 3t - 2 \\ y = 2t - 1 \\ z = t + 1 \end{cases}$$

   Which of the following numbers is the distance from α to r?

   (a) 0.
   (b) 4/3.
   (c) 8/3.
   (d) 7/3.

3. Let $\mathbb{R}^{3,3}$ be the real vector space of matrices with 3 rows and 3 columns, and let $\mathbb{R}_4[x]$ be the real vector space of polynomials in the variable x, of degree at most 4.

   Let $f : \mathbb{R}^{3,3} \to \mathbb{R}_4[x]$ be a linear map.

   Which of the following statements is true?

   (a) $\dim\big(\mathrm{Ker}(f)\big) < 3$.
   (b) $\dim\big(\mathrm{Ker}(f)\big) \geq 4$.
   (c) $\dim\big(\mathrm{Im}(f)\big) = \dim\big(\mathrm{Ker}(f)\big)$.
   (d) None of the other statements is true.

4. Consider the following linear systems in 4 unknowns:

$$S : x + y + z + t = 1 \qquad \text{e} \qquad S' : \begin{cases} x - y + z & = & 1 \\ 2y + t & = & 1 \end{cases}$$

Let $sol(S)$ and $sol(S')$ the set of (real) solutions of S and S' respectively. Which of the following statements is true?

(a) $sol(S) \cap sol(S') = \emptyset$.
(b) $sol(S) = sol(S')$.
(c) $sol(S')$ is strictly contained in $sol(S)$.
(d) $sol(S)$ is strictly contained in $sol(S')$.

5. Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = (1, 2, 3, 1)\,, \quad v_2 = (0, 1, 1, 2)\,, \quad v_3 = (-1, 2, 0, 1)$$

Which of the following statements is true?

(a) $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^4$.
(b) The vector subspace generated by $\{v_1, v_2, v_3\}$ has dimension 2.
(c) The vector subspace generated by $\{v_1, v_2, v_3\}$ has dimension 1.
(d) $\{v_1, v_2, v_3\}$ is a set of linearly independent vectors.

6. Let f be an endomorphism of $\mathbb{R}^4$ and let

$$\begin{pmatrix} 0 & 0 & 3 & 2 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

be the matrix representing f with respect to the basis (e_1, e_2, e_3, e_4) of $\mathbb{R}^4$, where

$$e_1 = (1, 0, 0, 0)\,,\ e_2 = (0, 1, 0, 0)\,,\ e_3 = (0, 0, 1, 0)\,,\ e_4 = (0, 0, 0, 1)\,.$$

Which of the following statements is true?

(a) e_1 is an eigenvector for f.
(b) e_3 is an eigenvector for f.
(c) e_4 is an eigenvector for f.
(d) e_2 is an eigenvector for f.

7. Find the positive definite quadratic form.

   (a) $x^2 - y^2 + 100xy$.
   (b) $2x^2 + 3y^2 - 2xy$.
   (c) $2x^2 + y^2 + 6xy$.
   (d) $-x^2 - y^2 + 100xy$.

8. Let $\mathbb{R}^{4,4}$ be the real vector space of matrices with 4 rows and 4 columns. Let $A \in \mathbb{R}^{4,4}$ and $B \in \mathbb{R}^{4,4}$.

   If $\text{rk}(A) = 3$ and $\text{rk}(B) = 4$ then:

   (a) $\text{rk}(AB) = 4$.
   (b) $\text{rk}(AB) = 3$.
   (c) $\text{rk}(AB) = 0$.
   (d) $\text{rk}(AB) = 1$.

EXERCISE

Let $\mathbb{R}[x]_3$ be the vector space of polynomials with real coefficients of degree at most 3 in the variable x. Consider the endomorphism $f\colon \mathbb{R}[x]_3 \to \mathbb{R}[x]_3$ defined by

$$f(a + bx + cx^2 + dx^3) = \frac{d^2}{dx^2}(a + bx + cx^2 + dx^3),$$

where $\frac{d^2}{dx^2}$ denotes the second derivative with respect to x.

(a) Compute the matrix $M_{\mathcal{B}}^{\mathcal{B}}(f)$ associated to the endomorphism f with respect to the basis $\mathcal{B} = (x^3, 1, x, x^2)$ of $\mathbb{R}[x]_3$.

(b) Compute $\dim(\mathrm{Im}(f))$ and find a subset of $\mathbb{R}[x]_3$ that makes up a basis of $\mathrm{Im}(f)$.

(c) Show that the polynomial

$$p(x) = \frac{18}{19} - 3x$$

is an eigenvector of f. Compute a basis of the eigenspace of f containing $p(x)$.

(d) Determine all eigenvalues of the matrix of f, together with their algebraic and geometric multiplicities.

(e) Establish, giving reasons for your answer, whether f is diagonalizable.

Solutions.

QUIZZES:

1	2	3	4	5	6	7	8
c	a	b	a	d	d	b	b

EXERCISE:

(a) Let's call the elements of the basis $\mathcal{B} = (p_1 = x^3, p_2 = 1, p_3 = x, p_4 = x^2)$ (which is just a permutation of the elements of the standard basis). As usual, we know that

$$M_{\mathcal{B}}^{\mathcal{B}}(f) = \left([f(p_1)]_{\mathcal{B}} \,\middle|\, [f(p_2)]_{\mathcal{B}} \,\middle|\, [f(p_3)]_{\mathcal{B}} \,\middle|\, [f(p_4)]_{\mathcal{B}} \right),$$

hence we compute

$$\begin{aligned}
f(p_1) &= \frac{d^2}{dx^2}x^3 = 6x \quad \Rightarrow \quad [f(p_1)]_{\mathcal{B}} = (0,0,6,0),\\
f(p_2) &= \frac{d^2}{dx^2}1 = 0 \quad \Rightarrow \quad [f(p_2)]_{\mathcal{B}} = (0,0,0,0),\\
f(p_3) &= \frac{d^2}{dx^2}x = 0 \quad \Rightarrow \quad [f(p_3)]_{\mathcal{B}} = (0,0,0,0),\\
f(p_4) &= \frac{d^2}{dx^2}x^2 = 2 \quad \Rightarrow \quad [f(p_1)]_{\mathcal{B}} = (0,2,0,0),
\end{aligned}$$

e hence

$$A = M_{\mathcal{B}}^{\mathcal{B}}(f) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b) The image of f is the column space of A, and it's easy to see that it has dimension 2 and a basis for it has coordinates $(0,0,6,0)$ and $(0,2,0,0)$, that is, a basis of $\mathrm{Im}(f)$ is given by the subset $(6x, 2)$ of $\mathbb{R}[x]_3$.

(c) The polynomial $p(x) = \frac{18}{19} - 3x \in \mathrm{Ker}(f)$ is an element of the kernel of f, and hence it is an eigenvector relative to the eigenvalue 0 (with respect to any basis). We compute a basis of the eigenspace relative to the eigenvalue 0 using the basis $\mathcal{B}$, and hence the matrix $A = M_{\mathcal{B}}^{\mathcal{B}}(f)$. Remark that, if $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, its coordinates with respect to the basis $\mathcal{B}$ are trivially $[p(x)]_{\mathcal{B}} = (a_3, a_0, a_1, a_2)$, so:

$$\begin{aligned}
E_A(0) = \mathrm{Ker}(A) &= \left\{ p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in \mathbb{R}[x]_3 \,\middle|\, A \cdot [p(x)]_{\mathcal{B}} = 0 \right\}\\
&= \left\{ p(x) \in \mathbb{R}[x]_3 \,\middle|\, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_3 \\ a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}\\
&= \left\{ p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in \mathbb{R}[x]_3 \,\middle|\, a_2 = a_3 = 0 \right\}.
\end{aligned}$$

(After all, we are simply saying that the polynomials whose second derivative is zero are those of degree ≤ 2.) To conclude, a basis of the eigenspace containing $p(x)$, which is just the kernel of the endomorphism, is given by the elements $(1, x)$.

(d) Let us continue to compute everything with respect to the basis $\mathcal{B}$: we already showed that $\lambda = 0$ is an eigenvalue with $m_g(0) = 2$. Moreover:

$$\det(A - tI_4) = \det \begin{pmatrix} -t & 0 & 0 & 0 \\ 0 & -t & 0 & 2 \\ 6 & 0 & -t & 0 \\ 0 & 0 & 0 & -t \end{pmatrix} = t^4,$$

that is, 0 is the only eigenvalue, with $m_a(0) = 4$ and $m_g(0) = 2$.

(e) From the previous part, it follows that f is not diagonalizable.