Linear algebra and geometry, a.y. 2024-2025

LAG's exam simulation, written test: 8 multiple choice quizzes and 1 exercise

DIRECTIONS: this is a simulation of the written test for Linear algebra and geometry. **Since it is a simulation, it should not be considered as representative of the program of the course, neither it should be an indication of the subjects that could appear in the examination.** If you have any questions send me an email and ask.

Let me also remind you that:

- You are admitted to the written test only after passing the computer based test on the numerical analysis part, that is after obtaining at least 4.25 (out of 10) points in the first test.
- The written test concerns topics from the theoretical/geometry part, and it is worth a total of 23 points: 2 points for each quiz + a maximum of 7 points for the exercise. There is no penalty for wrong quizzes. You need to get at least 7 points in the quizzes part and at least 2 points in the exercise to pass the written test.
- The final grade is the sum of the points obtained in the two tests. The exam is passed if the grade is at least 18/30; the "lode" is given to whomever obtains 33/30.
- You can always check the exam rules on the teaching portal by clicking here, or reading my exam handbook.

QUIZZES

1. Consider the plane
$$\alpha : \ x + 2y + 2z = 0$$
and the line
$$r : \begin{cases} x = -2t + 1 \\ y = t + 2 \\ z = 1 \end{cases}$$
Which of the following numbers is the distance from α to r?
   (a) 4/3.
   (b) 7/3.
   (c) 0.
   (d) 5/3.

2. Let us fix a cartesian system of coordinates Oxy in the plane. Let h be a real parameter, and let us consider the conic $\mathcal{C}_h$ defined by the following equation:
$$-2hx^2 + 2hy^2 + 4xy + 2x - 2y + 1 = 0.$$
Which of the following statements is true?
   (a) There is no $h \in \mathbb{R}$ such that $\mathcal{C}_h$ is degenerate.
   (b) For $h = \sqrt{2}$, $\mathcal{C}_h$ is degenerate.
   (c) $\mathcal{C}_h$ is a parabola, for all values of h.
   (d) $\mathcal{C}_h$ is an ellipse, for all values of h.

3. Let
$$A = \begin{pmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 6 & 6 \\ 0 & 1 & 6 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$
Which of the following statements is true?
   (a) $\det(A + B) = \det(A \cdot B)$.
   (b) $\det(A + B) = \det(A)$.
   (c) $\det(A + B) = \det(B)$.
   (d) $\det(A + B) = \det(A) + \det(B)$.

4. Let $\vec{w} \in \mathbb{R}^3$ be a unit vector. Consider the endomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$f(\vec{v}) = \vec{v} - (\vec{v} \cdot \vec{w})\vec{w}\,.$$

Which of the following statements is true?

(a) f is surjective.
(b) f is injective.
(c) $\vec{w}$ does not belong to $\mathrm{Ker}(f)$.
(d) f is neither injective nor surjective.

5. Let $\mathbb{R}_2[x]$ be the real vector space of polynomials in the variable x, of degree at most 2. Let h be a real parameter. Consider the polynomials

$$p_1(x) = 1 + x + x^2, \quad p_2(x) = 1 + 2x^2, \quad p_3(x) = x + hx^2.$$

Which of the following statements is true?

(a) $p_1(x)$, $p_2(x)$, $p_3(x)$ form a basis of $\mathbb{R}_2[x]$ for all h.
(b) $p_1(x)$, $p_2(x)$, $p_3(x)$ are a system of generators $\mathbb{R}_2[x]$ for $h = -1$.
(c) $p_1(x)$, $p_2(x)$, $p_3(x)$ are linearly dependent for exactly one value of h.
(d) $p_1(x)$, $p_2(x)$, $p_3(x)$ are linearly independent for all values of h.

6. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Which of the following statements is true?

(a) A has two distinct eigenspaces of dimension 1.
(b) A has rank 1.
(c) A has all eigenvalues of algebraic multiplicity 1.
(d) A is diagonalizable.

7. Given

$$q(x,y,z) = (x,y,z)\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

find the correct statement.

(a) The symmetric matrix associated to the quadratic form has only positive eigenvalues.
(b) $q(x,y,z) \geq 0$ for all $(x,y,z) \in \mathbb{R}^3$.
(c) There exists $(a,b,c) \in \mathbb{R}^3$ nonzero such that $q(a,b,c) = 0$.
(d) $q(x,x,x) > 0$ for all $x \in \mathbb{R}$.

8. Let A be a real 4×2 matrix and B a 2×4 real matrix and let $M = AB$. Consider the homogeneous linear system $MX = 0$.

Which of the following statements is true?

(a) The system has infinitely many solutions.
(b) The system has no solutions.
(c) The system has a unique solution.
(d) None of the other statements is true.

EXERCISE

Consider the endomorphism of $\mathbb{R}^4$ defined by:

$$f\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\qquad f\begin{pmatrix}1\\1\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\qquad f\begin{pmatrix}1\\1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\qquad f\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}.$$

(a) True or false? The matrix $M_{\mathcal{C}}^{\mathcal{C}}(f)$ associated to f with respect to the canonical basis $\mathcal{C} = (e_1, e_2, e_3, e_4)$ is

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Give reasons for your answer.

(b) Compute a basis of $\mathrm{Ker}(f)$.

(c) Compute a basis of $\mathrm{Im}(f)$.

(d) Determine all eigenvalues of f, together with their algebraic and geometric multiplicities.

(e) Compute a basis of the eigenspace $E_A(-1)$.

(f) Establish, giving reasons for your answer, whether it is true or false that A is diagonalizable.

Solutions.

QUIZZES:

1	2	3	4	5	6	7	8
b	a	d	d	c	a	c	a

EXERCISE:

(a) Since we know that

$$M_{\mathcal{C}}^{\mathcal{C}}(f) = \left(\begin{array}{c|c|c|c} [f(e_1)]_{\mathcal{C}} & [f(e_2)]_{\mathcal{C}} & [f(e_3)]_{\mathcal{C}} & [f(e_4)]_{\mathcal{C}} \end{array} \right),$$

we need to use the information that we are given to compute these coordinates. For the sake of simplicity, let's call

$$v_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It's easy to notice that

$$\begin{aligned} e_1 = v_4 \quad &\Rightarrow \quad f(e_1) = f(v_4), \\ e_2 = v_3 - v_4 \quad &\Rightarrow \quad f(e_2) = f(v_3 - v_4) = f(v_3) - f(v_4), \\ e_3 = v_2 - v_3 \quad &\Rightarrow \quad f(e_3) = f(v_2 - v_3) = f(v_2) - f(v_3), \\ e_4 = v_1 - v_2 \quad &\Rightarrow \quad f(e_4) = f(v_1 - v_2) = f(v_1) - f(v_2), \end{aligned}$$

and thus

$$M_{\mathcal{C}}^{\mathcal{C}}(f) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix} = A.$$

(b)

$$\text{Ker}(f) = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{cases} x - y = 0 \\ z - t = 0 \end{cases} \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{cases} x = x \\ y = x \\ z = z \\ t = z \end{cases} \right\} = \mathcal{L}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right).$$

(c) $\text{Im}(f)$ is the space generated by the columns of A, hence

$$\text{Im}(f) = \mathcal{L}\left(\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

(d) We already checked that $\dim(\mathrm{Ker}(f)) = 2$, so we expect the eigenvalue $\lambda = 0$ to have geometric multiplicity 2 and hence algebraic multiplicity ≥ 2. We compute

$$p_A(t) = \det \begin{pmatrix} 1-t & -1 & 0 & 0 \\ 0 & -t & 0 & 0 \\ -2 & 2 & -t & 0 \\ 1 & -1 & 1 & -1-t \end{pmatrix} = -t \cdot \det \begin{pmatrix} 1-t & 0 & 0 \\ -2 & -t & 0 \\ 1 & 1 & -1-t \end{pmatrix} = t^2(1-t)(-1-t),$$

hence $\lambda_1 = 0$, $m_a(0) = m_g(0) = 2$, $\lambda_2 = 1$, $m_a(1) = m_g(1) = 1$, $\lambda_3 = -1$, $m_a(-1) = m_g(-1) = 1$.

(e)

$$\begin{aligned} E_A(-1) &= \mathrm{Ker}(A + I_4) \\ &= \mathrm{Ker} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix} = \mathrm{Ker} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} = \mathrm{Ker} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{cases} x - y + z = 0 \\ y = 0 \\ z = 0 \end{cases} \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{cases} x = 0 \\ y = 0 \\ z = 0 \\ t = t \end{cases} \right\} = \mathcal{L} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

(f) Already from the computations of multiplicities in part (d) we knew that the endomorphism was diagonalizable.