Linear algebra and geometry, a.y. 2024-2025

LAG's exam simulation, written test: 8 multiple choice quizzes and 1 exercise

DIRECTIONS: this is a simulation of the written test for Linear algebra and geometry. **Since it is a simulation, it should not be considered as representative of the program of the course, neither it should be an indication of the subjects that could appear in the examination.** If you have any questions send me an email and ask.

Let me also remind you that:

- You are admitted to the written test only after passing the computer based test on the numerical analysis part, that is after obtaining at least 4.25 (out of 10) points in the first test.
- The written test concerns topics from the theoretical/geometry part, and it is worth a total of 23 points: 2 points for each quiz + a maximum of 7 points for the exercise. There is no penalty for wrong quizzes. You need to get at least 7 points in the quizzes part and at least 2 points in the exercise to pass the written test.
- The final grade is the sum of the points obtained in the two tests. The exam is passed if the grade is at least 18/30; the "lode" is given to whomever obtains 33/30.
- You can always check the exam rules on the teaching portal by clicking here, or reading my exam handbook.

1. Given $a \in \mathbb{R}$, consider the points

$$A = (a, 1, 4) \qquad B = (a, 1, 0) \qquad C = (a, 2, 1).$$

Which of the following statements is true?

(a) The area of the triangle ABC is 2 for all values of a.
(b) The area of the triangle ABC is $a^2/2$ for all values of a.
(c) The area of the triangle ABC is 4 for exactly one value of a.
(d) The points A, B, C are collinear for all values of a.

2. Given the basis $\mathcal{B} = \big((1, 2, 3), (1, 1, 1), (0, 2, 2)\big)$ of $\mathbb{R}^3$ and let $\vec{v} = (2, 5, 6)$. Which of the following statements is true?

(a) The components of $\vec{v}$ with respect to the basis $\mathcal{B}$ are $(2, 1, 1)$.
(b) The components of $\vec{v}$ with respect to the basis $\mathcal{B}$ are $(2, 5, 6)$.
(c) The components of $\vec{v}$ with respect to the basis $\mathcal{B}$ are $(1, 1, 1)$.
(d) The vector $\vec{v}$ is not linear combination of the vectors in $\mathcal{B}$.

3. Consider the real quadratic form

$$q(x, y) = (x, y)A \begin{pmatrix} x \\ y \end{pmatrix} = -3x^2 + 2xy - y^2,$$

where A is the real symmetric 2×2 matrix associated to the form $q(x, y)$. Which of the following statements is true?

(a) The matrix A has a zero eigenvalue.
(b) The characteristic polynomial of A is $t^2 + t + 6$.
(c) There does not exist $(a, b) \neq (0, 0)$ in $\mathbb{R}^2$ such that $q(a, b) = 0$.
(d) If $xy \neq 0$ then $q(x, y) > 0$.

4. Consider the lines

$$s: \quad x-1=y+z, \quad y+z=0,$$
$$r: \quad y+1=y+z, \quad y+z=0.$$

Which of the following numbers is the distance from r to s?

(a) 0.
(b) 3.
(c) $\sqrt{2}$.
(d) $\sqrt{3}$.

5. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & h \\ 0 & 2 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

where h is a real parameter.

Which of the following statements is true?

(a) A has 3 distinct eigenvalues for all values of $h \in \mathbb{R}$.
(b) If $h = 1$, the matrix A is not diagonalizable.
(c) A is diagonalizable only if $h = 0$.
(d) Se $h = 4$, A is diagonalizable.

6. Let $\mathbb{R}^{k,h}$ be the real vector space of matrices with k rows and h columns. Let $n > 1$ and $X \in \mathbb{R}^{n,1}$ nonzero.

Which of the following statements is true?

(a) There do not exist $A, B \in \mathbb{R}^{n,n}$ invertible and such that $AX = -BX$.
(b) If $A, B \in \mathbb{R}^{n,n}$ are such that $AX = -BX$, then $\text{rank}(A + B) < n$.
(c) For all pairs of matrices $A, B \in \mathbb{R}^{n,n}$ with $A \neq B$, one has $AX \neq -BX$.
(d) There exist $A, B \in \mathbb{R}^{n,n}$ such that $AX = -BX$ and $\text{rank}(A + B) = n$.

7. Consider the real quadratic form

$$q(x,y) = (x,y)A\begin{pmatrix} x \\ y \end{pmatrix} = x^2 - 13xy + 8y^2$$

where A is the 2×2 real symmetric matrix associated to $q(x,y)$.

Which of the following statements is true?

- There does not exist $(x_0, y_0) \in \mathbb{R}^2$ such that $q(x_0, y_0) < 0$.
- The equation $q(1,t) = 0$ does not have real solutions.
- The product of the eigenvalues of A is positive.
- $q(x_0, y_0) = 0$ for some $(x_0, y_0) \neq (0,0)$.

8. Let $A = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$, $D = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $E = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$.

Which of the following statements is true?

(a) BA is row-reduced.

(b) CA is row-reduced.

(c) DA is row-reduced.

(d) EA is row-reduced.

EXERCISE

Consider the endomorphism $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ such that $f(x, y, z, t) = (y, x, z, t)$.

(a) Compute the matrix $M^{\mathcal{C}}_{\mathcal{C}}(f)$ associated to f with respect to the canonical basis $\mathcal{C} = (e_1, e_2, e_3, e_4)$ both in the domain and codomain.

(b) Find a basis of the eigenspace relative to the eigenvalue -1.

(c) Prove or disprove: f is a diagonalizable endomorphism.

(d) Write down all the elements of the set $I = \{v \in \mathbb{R}^4 : f(v) = (1, 2, 3, 4)\}$.

(e) Given the basis $\mathcal{B} = (v_1, v_2, v_3, v_4)$ of $\mathbb{R}^4$, where

$$v_1 = (0, 0, 0, 2), \quad v_2 = (1, 0, 0, 2), \quad v_3 = (-1, -1, 0, 0), \quad v_4 = (0, 0, 1, -2),$$

Compute the matrix $M^{\mathcal{B}}_{\mathcal{B}}(f)$ associated to f with respect to the basis $\mathcal{B}$ both in the domain and codomain.

Solutions.

QUIZZES:

1	2	3	4	5	6	7	8
a	c	c	a	d	b	d	a

EXERCISE:

(a) We need to compute the 4×4 matrix whose columns are the images via f of the elements of the canonical basis with respect to the canonical basis itself, and hence:

$$M_{\mathcal{C}}^{\mathcal{C}}(f) = \left(\begin{array}{c|c|c|c} [f(e_1)]_{\mathcal{C}} & [f(e_2)]_{\mathcal{C}} & [f(e_3)]_{\mathcal{C}} & [f(e_4)]_{\mathcal{C}} \end{array} \right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) For the sake of simplicity, let's call $A = M_{\mathcal{C}}^{\mathcal{C}}$. The eigenspace we are looking for is:

$$E_A(-1) = \operatorname{Ker}(A + I_4) = \operatorname{Ker} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{cases} x = -y \\ y = y \\ z = 0 \\ t = 0 \end{cases} \right\} = \mathcal{L} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

(c) We compute the other eigenvalues of f and their multiplicities: since

$$p_A(t) = \det(A - tI_4) = \det \begin{pmatrix} -t & 1 & 0 & 0 \\ 1 & -t & 0 & 0 \\ 0 & 0 & 1-t & 0 \\ 0 & 0 & 0 & 1-t \end{pmatrix} = (1-t)^2(t^2-1) = (1-t)^3(-t-1),$$

$\lambda = -1$ is an eigenvalue with $m_a(-1) = 1$, hence $m_g(-1) = 1$. Moreover there is another eigenvalue $\mu = 1$ with $m_a(1) = 3$. Let us compute

$$m_g(1) = \dim E_A(1) = 4 - \operatorname{rk}(A - I_4) = 4 - \operatorname{rk} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 4 - 1 = 3.$$

We deduce that the endomorphism is diagonalizable.

(d) The elements of the set I are the solutions of the linear system

$$A \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} y \\ x \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix},$$

that has $\infty^{4-4} = \infty^0 =$ a unique solution $(x = 2, y = 1, z = 3, t = 4)$.

(e) As in part (a), we need to compute

$$M^{\mathcal{B}}_{\mathcal{B}}(f) = \left(\begin{array}{c|c|c|c} [f(v_1)]_{\mathcal{B}} & [f(v_2)]_{\mathcal{B}} & [f(v_3)]_{\mathcal{B}} & [f(v_4)]_{\mathcal{B}} \end{array} \right).$$

We can use the matrix of change of basis:

$$M^{\mathcal{B}}_{\mathcal{B}}(f) = M^{\mathcal{C}}_{\mathcal{B}}(id) \cdot M^{\mathcal{C}}_{\mathcal{C}}(f) \cdot M^{\mathcal{B}}_{\mathcal{C}}(id) = (M^{\mathcal{B}}_{\mathcal{C}}(id))^{-1} \cdot M^{\mathcal{C}}_{\mathcal{C}}(f) \cdot M^{\mathcal{B}}_{\mathcal{C}}(id).$$

One has that

$$M^{\mathcal{B}}_{\mathcal{C}}(id) = \left(\begin{array}{c|c|c|c} [v_1]_{\mathcal{C}} & [v_2]_{\mathcal{C}} & [v_3]_{\mathcal{C}} & [v_4]_{\mathcal{C}} \end{array} \right) = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & -2 \end{pmatrix},$$

hence we compute

$$\begin{aligned} M^{\mathcal{B}}_{\mathcal{B}}(f) &= \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & 0 & -2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & 0 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 & 1 & 1/2 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Alternatively (and probably in this case it's faster), we can compute explicitly the images via f of the elements of the basis $\mathcal{B}$, and then their coordinates with respect to $\mathcal{B}$ itself.

Let's start with the easy ones:

$$\begin{aligned} f(v_1) = (0,0,0,2) = v_1 \quad &\Rightarrow \quad [f(v_1)]_{\mathcal{B}} = (1,0,0,0), \\ f(v_3) = (-1,-1,0,0) = v_3 \quad &\Rightarrow \quad [f(v_3)]_{\mathcal{B}} = (0,0,1,0), \\ f(v_4) = (0,0,1,-2) = v_4 \quad &\Rightarrow \quad [f(v_4)]_{\mathcal{B}} = (0,0,0,1). \end{aligned}$$

The only one that requires a small computation is $f(v_2) = (0,1,0,2)$, because we need to find $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that:

$$f(v_2) = (0,1,0,2) = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 = (\lambda_2 - \lambda_3, -\lambda_3, \lambda_4, 2\lambda_1 + 2\lambda_2 - 2\lambda_4)$$

which means $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = 0$, so we find again the matrix

$$M^{\mathcal{B}}_{\mathcal{B}}(f) = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$