Linear algebra and geometry a.y. 2024-2025

Mixed quizzes on inner products, orthogonal diagonalization, quadratic forms, conics, spheres & circles

1. Let $q(x, y) = x^2 - 8xy - y^2$ be a quadratic form.

   Which of the following statements is true?

   (a) q is positive definite.

   (b) The matrix associated to q is $\begin{pmatrix} 1 & -4 \\ -4 & -1 \end{pmatrix}$.

   (c) q is negative definite.

   (d) The matrix associated to q is $\begin{pmatrix} -1 & -4 \\ -4 & 1 \end{pmatrix}$.

2. Given the polynomial

$$p(t) = (t^2 + t + 1)(t^2 - 2t + 1)\,,$$

   which of the following statements is true?

   (a) There exists a symmetric positive definite matrix $A \in \mathbb{R}^{4,4}$ having $p(t)$ as characteristic polynomial.

   (b) If a matrix A has $p(t)$ as characteristic polynomial, then A has a 3-dimensional eigenspace.

   (c) None of the other statements is true.

   (d) There is no real symmetric matrix whose characteristic polynomial is $p(t)$.

3. Consider the quadratic form

$$q(x, y, z) = (x, y, z) \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

   Which of the following statements is true?

   (a) The matrix associated to $q(x, y, z)$ has nonzero determinant.

   (b) There exists $(0, 0, 0) \neq (a, b, c) \in \mathbb{R}^3$ such that $q(a, b, c) = 0$.

   (c) $q(x, y, z)$ is positive definite.

   (d) $Q(x, y) = q(x, y, y)$ is positive definite.

4. Let A be a real symmetric 5×5 matrix with zero trace and zero determinant.

   Which of the following statements is true?

   (a) 0 is an eigenvalue of A.
   (b) A is (positive or negative) definite.
   (c) A is (positive or negative) semidefinite.
   (d) None of the other statements is true.

5. Let A be a real 3×3 symmetric matrix having an eigenvalue equal to 1. If the eigenspace $E_A(1)$ relative to the eigenvalue 1 is the set of vectors (x, y, z) such that $x + y - z = 0$, which of the following statements is true?

   (a) A is not diagonalizable.
   (b) $(1, 1, -1)$ is an eigenvector of A relative to the eigenvalue 1.
   (c) $(1, 1, -1)$ is not an eigenvector of A.
   (d) $(1, 1, -1)$ is an eigenvector of A relative to an eigenvalue different from 1.

6. Consider the quadratic form with real coefficients

$$q(x, y, z) = 10x^2 + 4y^2 + 4yz + z^2.$$

   Which of the following statements is true?

   (a) There exists $(a, b, c) \in \mathbb{R}^3$ such that $q(a, b, c) > 0$.
   (b) None of the other statements is true.
   (c) There is no $(a, b, c) \in \mathbb{R}^3$ with $(a, b, c) \neq (0, 0, 0)$ such that $q(a, b, c) = 0$.
   (d) For all $(a, b, c) \in \mathbb{R}^3$, $q(a, b, c) \leq 0$.

7. Given the quadratic form

$$f(x, y) = (x, y)A \begin{pmatrix} x \\ y \end{pmatrix} = x^2 - 3xy + 8y^2,$$

   which of the following statements is true?

   (a) The determinant of the matrix A is a negative number.
   (b) There exists $(a, b) \in \mathbb{R}^2$ such that $f(a, b) < 0$.
   (c) If $xy \neq 0$, then $f(x, y) > 0$.
   (d) The determinant of the matrix A cannot be computed.

8. Consider the quadratic form

$$q(x,y,z) = (x,y,z)B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -x^2 + 2y^2 + 2xz + z^2.$$

Which of the following statements is true?

(a) The matrix B admits both positive and negative eigenvalues.
(b) $q(x,y,x+y) = 0$.
(c) $q(y+z,y,z) = 0$.
(d) The matrix B has rank 2.

9. Let h be a real parameter, and consider in the Euclidean plane the family of conics described by the equation

$$x^2 + 12xy + 11y^2 + h - 1 = 0.$$

Which of the following statements is true?

(a) $\mathcal{C}_h$ is a parabola, for all values of h.
(b) $\mathcal{C}_h$ is an ellipse, for all values of h.
(c) When $h = 1$, the conic is degenerate.
(d) None of the other statements is true.

10. In the Euclidean plane with a fixed coordinate system Oxy, consider the conic $\mathcal{C}$ of equation $2x^2 + 5y^2 - 2xy = 0$.

Which of the following statements is true?

(a) $\mathcal{C}$ is a double line.
(b) $\mathcal{C}$ is a parabola.
(c) $\mathcal{C}$ is non degenerate.
(d) $\mathcal{C}$ is the union of two lines.

11. Find the negative definite quadratic form.

(a) $x^2 + y^2 - 100xy$.
(b) $y^2 - x^2 - 100xy$.
(c) $2xy - 3x^2 - 2y^2$.
(d) $-2x^2 - y^2 - 6xy$.

12. Consider the family of conics described by the equation

$$x^2 + kxy + y^2 + kx - 1 = 0,$$

where $k \in \mathbb{R}$ is a real parameter.

Find the true statement.

(a) There exists precisely two values of k such that the equation represents a hyperbola.
(b) There is no value of k such that the equation represents a degenerate conic.
(c) There is no value of k such that the equation represents a parabola.
(d) None of the other statements is true.

13. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, and let $p_A(t)$ be its characteristic polynomial.

Find the true statement.

(a) $p_A(t) = t^4 + t^3 + t^2 + t + 1$.
(b) $p_A(t) = t^4 + t^3 + t$.
(c) $p_A(t) = t^4 - 4t^3$.
(d) $p_A(t) = -t^4 + t^3 + t^2 + t$.

14. Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 10 \end{pmatrix} \in \mathbb{R}^{3,3}.$$

Which of the following statements is true?

(a) A is positive semidefinite.
(b) A does not have positive eigenvalues.
(c) A does not have a 0 eigenvalue.
(d) None of the other statements is true.

15. Given the two circles

$$\mathcal{C}_1 : \{x^2 + y^2 + z^2 - 1 = x + y = 0\} \qquad \text{and} \qquad \mathcal{C}_2 : \{x^2 + y^2 + z^2 - 1 = y = 0\},$$

find the true statement.

(a) $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.
(b) $\mathcal{C}_1 \cap \mathcal{C}_2$ contains precisely 1 point.
(c) $\mathcal{C}_1 \cap \mathcal{C}_2$ contains more than 1 point.
(d) $\mathcal{C}_1 = \mathcal{C}_2$.

16. The two circles

$$\mathcal{C}_1 : \{x^2 + y^2 + z^2 - 100 = x + 2z - 1 = 0\} \quad \text{and} \quad \mathcal{C}_2 : \{x^2 + y^2 + z^2 - 100 = x + 2z - 2 = 0\}$$

(a) have the same center and the same radius.
(b) have the same center and different radiuses.
(c) have different centers and the same radius.
(d) have different centers and different radiuses.

17. Let $\mathcal{S} : \{x^2+y^2+z^2-1 = 0\}$ be a sphere with radius R, and $\mathcal{C} : \{x^2+y^2+z^2+y-1 = 0 = y\}$ be a circle with radius r.

Find the true statement.

(a) $r > R$.
(b) $r = R$.
(c) $r < R$.
(d) $\mathcal{S}$ does not contain $\mathcal{C}$.

18. In which of the following cases the circle $\mathcal{C} = \pi \cap S$ has radius $\sqrt{3}$ and center $(1, -1, 2)$?

(a) $S : x^2 + y^2 + z^2 - 2x + 2y - 4z = -3, \qquad \pi : x - y + 2z = 6.$
(b) $S : x^2 + y^2 + z^2 - 2x + 2y - 4z = 3, \qquad \pi : x - y + 2z = 6.$
(c) $S : x^2 + y^2 + z^2 - 2x + 2y - 4z = -3, \qquad \pi : x + y + 2z = 6.$
(d) $S : x^2 + y^2 + z^2 + 2x + 2y - 4z = -3, \qquad \pi : x - y + 2z = 6.$

Solutions

1. (b)
2. (d)
3. (b)
4. (a)
5. (d)
6. (a)
7. (c)
8. (a)
9. (c)
10. (d)
11. (c)
12. (b)
13. (c)
14. (a)
15. (c)
16. (d)
17. (b)
18. (a)