

Mixed quizzes on linear maps, associated matrices, diagonalization

1. Let A, B be two $n \times n$ matrices such that $0 < \text{rk}(A) < \text{rk}(B) = n$.

Find the true statement.

- (a) $\text{rk}(AB) = n\text{rk}(B)$
- (b) AB is invertible
- (c) $\det(AB) = \det(A)$
- (d) $\lambda = 0$ is not an eigenvalue of AB

2. Let $A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$.

Find the true statement.

- (a) $A^2 = A^3$
- (b) $\lambda = 4$ is not an eigenvalue of A^2
- (c) $(1, 1)$ is an eigenvector of A
- (d) $\det(A^3) = 8$

3. Let $A \in \mathbb{R}^{4,5}$ and $B \in \mathbb{R}^{5,4}$ be two matrices, and consider the endomorphism $f : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ associated to the product BA .

Find the true statement.

- (a) f is not an isomorphism
- (b) f is injective but not surjective
- (c) $\det(BA) \neq 0$
- (d) None of the other statements is true.

4. Consider the linear map $f(x, y, z) = (x + y + z, 0, 0)$.

Find the true statement.

- (a) f is surjective
- (b) f is not injective
- (c) $(1, 1, 1)$ is sent by f into a multiple of itself.
- (d) $\dim \text{Ker}(f) = 3$

5. Consider the linear map $f : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ defined by $f(x, y, z, t) = (x + y, z + t)$.

Find the true statement.

- (a) f is injective
- (b) f is surjective
- (c) $f^{-1}(2, 2) = \{(1, 1, 1, 1)\}$
- (d) $(1, 1, 1, 1) \in \text{Ker}(f)$

6. Let $\mathbb{R}[x]_3$ be the vector space of polynomials in the variable x with real coefficients and degree ≤ 3 and consider the linear map $f : \mathbb{R}[x]_3 \rightarrow \mathbb{R}[x]_3$ such that

$$f(p(x)) = p(0)x^2.$$

Find the true statement.

- (a) f is injective
- (b) f is an isomorphism
- (c) $\dim(\text{Ker}(f)) = 2$
- (d) $\dim(\text{Im}(f)) = 1$

7. Let A be an $n \times n$ matrix having eigenvalue $\lambda = 1$ with algebraic multiplicity $m_a(1) = n$.

Find the true statement.

- (a) A power of $A - I_n$ is the zero matrix.
- (b) A is not invertible
- (c) A power of $A + I_n$ is the zero matrix
- (d) $A - I_n$ is invertible

8. Consider the linear map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $f(1, 2) = (2, 1)$ and $f(2, 1) = (1, 2)$.

Find the true statement.

- (a) f is not surjective
- (b) $f^{-1}(3, 3) = \{(1, 1)\}$
- (c) f is injective
- (d) $\{(0, -1), (1, 0)\}$ is a set of generators of $\text{Ker}(f)$

9. Let $\mathbb{R}[x]_2$ be the vector space of polynomials in x , with real coefficients and degree ≤ 2 . Define the endomorphism $f: \mathbb{R}_2[x] \rightarrow \mathbb{R}_2[x]$ as:

$$f(ax^2 + bx + c) = cx^2 - b,$$

and let $M_{\mathcal{B}}^{\mathcal{B}}(f)$ be the matrix associated to f with respect to the basis $\mathcal{B} = (x^2, x, 1)$ (both in the domain and codomain).

Which of the following statements is true?

(a) $M_{\mathcal{B}}^{\mathcal{B}}(f) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(b) $M_{\mathcal{B}}^{\mathcal{B}}(f) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$

(c) $M_{\mathcal{B}}^{\mathcal{B}}(f) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$

(d) $M_{\mathcal{B}}^{\mathcal{B}}(f) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

10. Let f be an endomorphism $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $f(1, 1, 0) = f(1, 0, 1) = (0, 1, 1)$.

Which of the following statements is true?

- (a) None of the other statements is true.
- (b) f is surjective.
- (c) $\dim(\text{Ker } f) \geq 1$.
- (d) f is injective.

11. Let $\vec{w} \in \mathbb{R}^3$ be a unit vector. Consider the endomorphism $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by:

$$f(\vec{v}) = \vec{v} - (\vec{v} \cdot \vec{w}) \vec{w}.$$

Which of the following statements is true?

- (a) f is neither injective nor surjective.
- (b) f is surjective.
- (c) f is injective.
- (d) \vec{w} does not belong to $\text{Ker}(f)$.

12. Let $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be an endomorphism with characteristic polynomial

$$p_f(t) = (t^2 - 7)(t^2 - 4t).$$

Which of the following statements is true?

- (a) $\dim(\text{Ker}(f)) = 2$
- (b) $\dim(\text{Im}(f)) = 4$
- (c) f is injective
- (d) $\dim(\text{Im}(f)) = 3$

13. Given a linear map $f : \mathbb{R}^4 \rightarrow \mathbb{R}^2$, which of the following statements is surely false?

- (a) $\dim \text{Ker}(f) = 2$.
- (b) f is surjective.
- (c) $\dim \text{Ker}(f) = 1$.
- (d) The matrix associated to f has rank 2.

14. Consider the endomorphism $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by:

$$f(x, y, z) = (x + z, 2y, x + z).$$

Which of the following statements is true?

- (a) f is an isomorphism.
- (b) f is diagonalizable.
- (c) f is not simple.
- (d) All eigenspaces of f have dimension 1.

15. Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

which of the following statements is true?

- (a) The characteristic polynomial of A is $-t^3 + 6t^2$.
- (b) The characteristic polynomial of A is $-t^3 + 6t$.
- (c) The characteristic polynomial of A is $-t^3 + 6$.
- (d) The characteristic polynomial of A is $t^3 + 6t^2$.

16. Given the endomorphism $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with characteristic polynomial $-(t-1)t^2$, which of the following statements is true?

- (a) If the image of f has dimension 1, then f is diagonalizable.
- (b) f has 3 linearly independent eigenvectors.
- (c) 4 is an eigenvalue of f .
- (d) If $\dim(\text{Ker}(f)) = 1$, then f is diagonalizable.

17. Let $V = \{(x, y, z) \in \mathbb{R}^3 : x + y - z = 0\}$ and let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be an endomorphism such that $V = \text{Ker}(f)$ and 1 is an eigenvalue of f .

Which of the following statements is true?

- (a) f has 3 distinct eigenvalues.
- (b) $\dim(\text{Im}(f)) = 2$.
- (c) f is diagonalizable.
- (d) The characteristic polynomial of f could be $-t(t-1)^2$.

18. Let g be an endomorphism of \mathbb{R}^3 and let

$$M_{\mathcal{C}}^{\mathcal{C}}(g) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 4 & 1 \\ 5 & 0 & 3 \end{pmatrix}$$

be the matrix associated to g with respect to the canonical basis $\mathcal{C} = (e_1, e_2, e_3)$ of \mathbb{R}^3 .

Which of the following statements is true?

- (a) e_1 is an eigenvector of g .
- (b) e_3 is an eigenvector of g .
- (c) None of the other statements is true.
- (d) e_2 is an eigenvector of g .