Linear algebra and geometry a.y. 2024-2025

Worksheet 15: exercises on chapters 22–24 from the lecture notes

(Some of these exercises come from the books by [Schlesinger], [Baldovino-Lanza], [Sernesi], [Leon])

1. Consider the following pencil of conics:
$$\mathcal{C}_t : \quad x^2 + (1-t)y^2 + 2tx - 2(1-t)y + 2 - t = 0,$$
and find the values of the parameter t such that $\mathcal{C}_t$ is a
   (a) parabola;
   (b) hyperbola;
   (c) ellipse;
   (d) circle;
   (e) degenerate conic.

2. Classify the following conics:
   (a) $2x^2 + 2xy + x + 5y - 10 = 0$;
   (b) $3x^2 - 8xy - 3y^2 + 10 = 0$;
   (c) $9x^2 + 16y^2 + 24xy - 40x + 30y = 0$;
   (d) $3x^2 + 2xy + 3y^2 + 2\sqrt{2}x - 2\sqrt{2}y = 0$.

3. Find all degenerate conics in the family
$$\mathcal{F} : 2\alpha x^2 + 2y^2 + 4\alpha x + 2y + 2\alpha = 0,$$
where $\alpha \in \mathbb{R}$ is a real parameter.

4. Describe the conics in the family
$$x^2 - 4xy + y^2 + 7h^2 + 1 = 0$$
as the parameter $h \in \mathbb{R}$ varies.

5. Write down the equation of the sphere with center in the point $C = (0, 1, 1)$ and tangent to the plane $x + y + z = -1$.

6. Find the values of the real parameter k such that the plane of equation $x - 2y - z = k$ is tangent to the sphere $x^2 + y^2 + z^2 - 4y + 6z + 8 = 0$.

7. Write the equation of the circle with center in the point $C = (1, 1, 1)$, lying on a plane parallel to $\pi : \{2x - 3y + z + 2 = 0\}$, and with radius 3.

8. Write the equation of the circle passing through the points $O = (0, 0, 0)$, $P = (2, 0, 0)$ and $R = (0, 1, 0)$.

Solutions.

1. (a) Never;
   (b) $t > 1$;
   (c) $t < 1$, $t \neq -1$;
   (d) $t = 0$ (imaginary circle);
   (e) $t = \pm 1$.

2. (a) Two non parallel lines meeting in a point;
   (b) (equilateral) hyperbola;
   (c) parabola;
   (d) ellipse.

3. $y^2 + y = 0$ and $x^2 - 3y^2 + 2x - 3y + 1 = 0$.

4. The conic is a hyperbola for all values of $h \in \mathbb{R}$.

5. $x^2 + y^2 + z^2 - 2y - 2z - 1 = 0$

6. $k = -1 \pm \sqrt{30}$

7. $\begin{cases} x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0 \\ 2x - 3y + z = 0 \end{cases}$

8. $x^2 + y^2 - 2x - y = 0$

Please note. Remember that in general there might be more than one technique to solve the same exercise. If you find a typo, or something that you do not understand, let me know!