Linear algebra and geometry a.y. 2024-2025

Worksheet 14: exercises on chapters 21-22 from the lecture notes

1. Write down the symmetric matrix associated to each of the following real quadratic forms:
   (a) $q(x, y) = 3x^2 - 5xy + y^2$;
   (b) $q(x, y, z) = 2x^2 + 3y^2 + z^2 + xy - 2xz + 3yz$;
   (c) $q(x, y, z) = 33x^2 + 10y^2 + z^2 + xy + 8xz + 3yz$.

2. Find the character of definition of each of the quadratic forms in the previous exercise.

3. Discuss the character of definition of the quadratic form associated to the symnmetric matrix

$$\begin{pmatrix} 3 & -\alpha \\ -\alpha & 1 \end{pmatrix}$$

   as the parameter $\alpha \in \mathbb{R}$ varies.

4. Let $q(x, y, z)$ be the quadratic form associated to the symmetric matrix

$$\begin{pmatrix} 4 & 0 & -2 \\ 0 & -1 & 0 \\ -2 & 0 & 7 \end{pmatrix}.$$

   (a) What is the character of definition of q?
   (b) Find triples (x_1, y_1, z_1) and (x_2, y_2, z_2) with the property that $q(x_1, y_1, z_1)q(x_2, y_2, z_2) < 0$.

5. Verify that the equation $xy - 2x + y - 3 = 0$ represents an equilateral hyperbola in the plane, finding its canonical form and the rototranslation one should apply to get it.

Solutions.

1. (a) $\begin{pmatrix} 3 & -5/2 \\ -5/2 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 2 & 1/2 & -1 \\ 1/2 & 3 & 3/2 \\ -1 & 3/2 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 33 & 1/2 & 4 \\ 1/2 & 10 & 3/2 \\ 4 & 3/2 & 1 \end{pmatrix}$

2. (a) indefinite
   (b) indefinite
   (c) positive definite

3. Positive definite for $\alpha \in (-\sqrt{3}, \sqrt{3})$, indefinite for $\alpha < -\sqrt{3}$ and $\alpha > \sqrt{3}$, positive semidefinite for $\alpha = \pm\sqrt{3}$.

4. (a) Indefinite;
   (b) there are many such triples, for example $(1, 0, 0)$ and $(0, 1, 0)$.

5. The canonical equation of the conic is $\frac{x'^2}{2} - \frac{y'^2}{2} = 2$ which is precisely the canonical form of the equilateral hyperbola. This can be found by applying the rototranslation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} 1/\sqrt{2} \\ 3/\sqrt{2} \end{pmatrix}.$$

Please note. Remember that in general there might be more than one technique to solve the same exercise. If you find a typo, or something that you do not understand, let me know!