Linear algebra and geometry a.y. 2024-2025

Worksheet 13: exercises on chapter 20 from the lecture notes

1. On the vector space $\mathbb{R}^3$, let us define the following map:

$$\begin{aligned} \mathbb{R}^3 \times \mathbb{R}^3 &\to \mathbb{R} \\ (x, y) &\mapsto \frac{x_1 y_1}{4} + \frac{x_2 y_2}{2} + \frac{x_3 y_3}{4} = x \star y \end{aligned}$$

   (a) Verify that $x \star y$ defines an inner product on $\mathbb{R}^3$.
   (b) Show that $v = (1, 1, 1)$ and $w = (-5, 1, 3)$ are orthogonal with respect to $\star$.
   (c) Compute the lengths of v and w with respect to $\star$.

2. Which of the following applications are inner products on $\mathbb{R}^3$?

   (a) $\alpha : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $\quad \alpha(x, y) = x_1 y_2 + x_2 y_1 + x_3 y_3$;
   (b) $\beta : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $\quad \beta(x, y) = x_1 y_1 + x_2 y_2$;
   (c) $\gamma : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $\quad \gamma(x, y) = x_1 y_1 + x_2 y_2 + 3 x_3 y_3$;
   (d) $\delta : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $\quad \delta(x, y) = 2 x_1 y_1 - x_2 y_2 + x_3 y_3$.

3. Use the Gram-Schmidt algorithm to find an orthonormal (with respect to the standard inner product) basis of the plane $\pi : \{x + y - z = 0\}$ in $\mathbb{R}^3$.

4. Use the Gram-Schmidt algorithm to find an orthonormal (with respect to the standard inner product) basis in $\mathbb{R}^4$ of the row space of the matrix

$$A = \begin{pmatrix} 2 & 2 & 4 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix}.$$

5. Diagonalize the symmetric matrix

$$A = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \end{pmatrix},$$

   finding an orthonormal diagonalizing basis.

6. Let $A \in \mathbb{R}^{n,n}$ be a symmetric matrix, and let $\lambda_1 \neq \lambda_2$ two of its eigenvalues. Let v_1 and v_2 be eigenvectors relative to the eigenvalues λ_1 and λ_2 respectively. Try to prove that $v_1 \perp v_2$ with respect to the standard inner product in $\mathbb{R}^n$.

Solutions.

1. (a) Let $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3), z = (z_1, z_2, z_3) \in \mathbb{R}^3, \alpha \in \mathbb{R}$. The product $\star$ satisfies the following properties:

   - it is commutative, because

$$x \star y = \frac{x_1 y_1}{4} + \frac{x_2 y_2}{2} + \frac{x_3 y_3}{4} = \frac{y_1 x_1}{4} + \frac{y_2 x_2}{2} + \frac{y_3 x_3}{4} = y \star x;$$

   - it is distributive, because

$$\begin{aligned}(x+y) \star z &= \frac{(x_1 + y_1)z_1}{4} + \frac{(x_2 + y_2)z_2}{2} + \frac{(x_3 + y_3)z_3}{4} \\ &= \frac{x_1 z_1 + y_1 z_1}{4} + \frac{x_2 z_2 + y_2 z_2}{2} + \frac{x_3 z_3 + y_3 z_3}{4} \\ &= \frac{x_1 z_1}{4} + \frac{x_2 z_2}{2} + \frac{x_3 z_3}{4} + \frac{y_1 z_1}{4} + \frac{y_2 z_2}{2} + \frac{y_3 z_3}{4} \\ &= (x \star z) + (y \star z);\end{aligned}$$

   - it is compatible with the scalar multiplication, because

$$\alpha(x \star y) = \alpha\Big(\frac{x_1 y_1}{4} + \frac{x_2 y_2}{2} + \frac{x_3 y_3}{4}\Big) = \frac{\alpha x_1 y_1}{4} + \frac{\alpha x_2 y_2}{2} + \frac{\alpha x_3 y_3}{4} = (\alpha x) \star y;$$

   - it is positive definite, because

$$x \star x = \frac{x_1^2}{4} + \frac{x_2^2}{2} + \frac{x_3^2}{4} > 0 \text{ per ogni } x \neq 0_{\mathbb{R}^3}.$$

   All in all, $\star$ is an inner product on $\mathbb{R}^3$.

   (b) $v \star w = -\frac{5}{4} + \frac{1}{2} + \frac{3}{4} = 0$.

   (c) $|v|_\star = 1$, $|w|_\star = 3$.

2. (a) No;

   (b) no;

   (c) yes;

   (d) no.

3. An orthonormal basis of π is $\Big(u_1 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}), u_2 = (-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})\Big)$.

4. An orthonormal basis is $\Big(v_1 = (\frac{2}{5}, \frac{2}{5}, \frac{4}{5}, \frac{1}{5}), v_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0), v_3 = (-\frac{3}{5\sqrt{2}}, -\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, -\frac{4}{5\sqrt{2}})\Big)$.

5. $^tPAP = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, where $P = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$.

6. By definition, we know that $Av_1 = \lambda_1 v_1$ and $A\lambda_2 = \lambda_2 v_2$, and that $^tA = A$. Moreover, once we identify the elements of $\mathbb{R}^n$ with column matrices, the standard inner (dot) product is $v \cdot w = {}^tvw$, that is, row-column multiplication. Then

   $$\lambda_1(v_1 \cdot v_2) = (\lambda_1 v_1) \cdot v_2 = Av_1 \cdot v_2 = {}^t(Av_1)v_2 = {}^tv_1{}^tAv_2 = {}^tv_1(Av_2) = {}^tv_1(\lambda_2 v_2) = \lambda_2(v_1 \cdot v_2)$$

   and thus $(\lambda_1 - \lambda_2)(v_1 \cdot v_2) = 0$. Since we know that $\lambda_1 \neq \lambda_2$, the only possibility is $v_1 \cdot v_2 = 0$, which is exactly what we wanted to prove.

Please note. Remember that in general there might be more than one technique to solve the same exercise. If you find a typo, or something that you do not understand, let me know!