Linear algebra and geometry a.y. 2024-2025
Worksheet 12: exercises on chapters 18–19 from the lecture notes

(Some of these exercises come from the books by [Schlesinger], [Baldovino-Lanza], [Sernesi], [Leon])

1. Write down the matrix $M(f)$ associated to the endomorphism f with respect to the canonical basis, knowing that $f : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by the following properties:
   - $f(e_1) = 2e_1 - e_2 + 5e_3$;
   - e_2 is an eigenvector for f, relative to the eigenvalue $\lambda = -2$;
   - $f(e_2 + e_3) = -2e_1 + e_3$.

2. Write down the matrix $M(g)$ associated to the endomorphism g with respect to the canonical basis, knowing that $g : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by the following properties:
   - $v_1 = e_1 + e_2 + e_3$ is an eigenvector for g, relative to the eigenvalue $\lambda_1 = -1$;
   - $v_2 = 2e_1 + 3e_2$ is an eigenvector for g, relative to the eigenvalue $\lambda_2 = 3$;
   - $\mathrm{Ker}(g) = \{(x, y, z) \mid x = t, y = 2t, z = t, t \in \mathbb{R}\}$.

3. Find all eigenvalues and eigenvectors of the following matrices, and compute the algebraic and geometric multiplicities of the eigenvalues.

   (a) $A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$;

   (b) $B = \begin{pmatrix} 6 & -4 \\ 3 & -1 \end{pmatrix}$;

   (c) $C = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$;

   (d) $D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$;

   (e) $E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$;

   (f) $F = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}$.

4. Diagonalize, when possible, the following matrices over $\mathbb{R}$

   (g) $G = \begin{pmatrix} 2 & -8 \\ 1 & -4 \end{pmatrix}$;

   (h) $H = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$.

5. Diagonalize, if possible, matrices L and M over $\mathbb{R}$:

$$L = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{pmatrix}.$$

   Then use the diagonalization to compute L^6, M^4, L^{-1} and M^{-1}.

6. Try to prove that eigenvectors relative to distinct eigenvalues are linearly independent.

7. Let $A, B \in \mathbb{R}^{n,n}$ two matrices with the same diagonalizing matrix P. Try to prove that A and B commute, that is, $AB = BA$.

8. For which values of a and b in $\mathbb{R}$ is the following matrix $A_{a,b} \in \mathbb{R}^{4,4}$ diagonalizable?

$$A_{a,b} = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solutions.

1. $M_{\mathcal{C}}^{\mathcal{C}}(f) = \begin{pmatrix} 2 & 0 & -2 \\ -1 & -2 & 2 \\ 5 & 0 & 1 \end{pmatrix}$.

2. If we call v_3 the vector $v_3 = e_1 + 2e_2 + e_3$, then the three (lin. indep.) vectors v_1, v_2, v_3 form a basis for $\mathbb{R}^3$ made up by eigenvectors of g, let's call it $\mathcal{B}$. In other words:

$$M_{\mathcal{B}}^{\mathcal{B}}(g) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Then

$$M_{\mathcal{C}}^{\mathcal{C}}(g) = M_{\mathcal{C}}^{\mathcal{B}}(id) M_{\mathcal{B}}^{\mathcal{B}}(g) M_{\mathcal{B}}^{\mathcal{C}}(id) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3/2 & 1 & -7/2 \\ 3 & 1 & -5 \\ -3/2 & 1 & -1/2 \end{pmatrix}.$$

   Notice how the matrix written using the canonical basis is much more complicated than the one written using the basis of eigenvectors!

3. (a) $\lambda_1 = 5$, $E_A(5) = \mathcal{L}((1,1))$, $\lambda_2 = -1$, $E_A(-1) = \mathcal{L}((1,-2))$; all multiplicities are 1;

   (b) $\lambda_1 = 3$, $E_B(3) = \mathcal{L}((4,3))$, $\lambda_2 = 2$, $E_B(2) = \mathcal{L}((1,1))$; all multiplicities are 1;

   (c) $\lambda_1 = 2$, with $m_a(2) = 2$, $E_C(2) = \mathcal{L}((1,-1))$, thus $m_g(2) = 1$;

   (d) $\lambda_1 = 0$, $E_D(0) = \mathcal{L}((1,0,0))$; $m_a(0) = 3$, $m_g(0) = 1$;

   (e) $\lambda_1 = 2$, $E_E(2) = \mathcal{L}((1,1,0))$, $\lambda_2 = 1$, $E_E((1)) = \mathcal{L}((1,0,0),(0,1,-1))$; $m_a(2) = 1 = m_g(2)$, $m_a(1) = 2 = m_g(1)$;

   (f) $\lambda_1 = 1$, $E_F(1) = \mathcal{L}((1,0,0))$, $\lambda_2 = 4$, $E_F(4) = \mathcal{L}((1,1,1))$, $\lambda_3 = -2$, $E_F(-2) = \mathcal{L}((-1,-1,5))$; all multiplicities are 1.

4. (g) G has two distinct eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -2$, so it is diagonalizable.
   $E_G(0) = \mathcal{L}((4,1))$ and $E_G(-2) = \mathcal{L}((2,1))$, all multiplicities are 1.
   The diagonalizing matrix is $P = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$ and indeed

$$P^{-1}GP = \begin{pmatrix} 1/2 & -1 \\ -1/2 & 2 \end{pmatrix} \begin{pmatrix} 2 & -8 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

   (h) H only has one real eigenvalue $\lambda = 0$, therefore it is not diagonalizable over $\mathbb{R}$.

5. $P^{-1}LP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{pmatrix}$.

   $Q^{-1}MQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ where $Q = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & -1 \\ 2 & 1 & 1 \end{pmatrix}$.

   $L^6 = L$, $M^4 = \begin{pmatrix} 1 & 0 & 0 \\ 10 & 16 & 0 \\ -10 & 0 & 16 \end{pmatrix}$, L^{-1} does not exist ($\det(L) = 0$), $M^{-1} = \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 \\ -1 & 1 & 3 \\ 3 & 1 & -1 \end{pmatrix}$.

6. We can do it for 2 vectors, the general case is almost the same: let $\lambda \neq \mu$ be two distinct eigenvalues of an endomorphism $f : V \to V$, and let v and w two corresponding eigenvectors, that is, $f(v) = \lambda v$ and $f(w) = \mu w$. Suppose by contradiction that the two vectors are linearly dependent: then we would have $w = \alpha v$, where $\alpha \in K, \alpha \neq 0$. But then:

   $$w = \alpha v \ \Rightarrow\ f(w) = f(\alpha v) \ \Rightarrow\ \mu w = \alpha f(v) = \alpha\lambda v \ \Rightarrow\ \mu(\alpha v) = \lambda(\alpha v) \ \Rightarrow\ (\mu - \lambda)(\alpha v) = 0_V.$$

   By the zero product law, since $\mu - \lambda \neq 0$, and $v \neq 0_V$, necessarily $\alpha v = 0_V$, which is a contradiction.

7. As a first remark, notice that diagonal matrices commute: if $D_1 = Diag(\lambda_1, \ldots, \lambda_n)$ and $D_2 = Diag(\mu_1, \ldots, \mu_n)$ are two diagonal matrices, one has that:

   $$D_1D_2 = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} = \begin{pmatrix} \lambda_1\mu_1 & & \\ & \ddots & \\ & & \lambda_n\mu_n \end{pmatrix} = D_2D_1.$$

   Let now $D_1 = P^{-1}AP$ and $D_2 = P^{-1}BP$ be the two diagonal matrices similar to A and B respectively. The we can compute

   $$AB = (PD_1P^{-1})(PD_2P^{-1}) = PD_1D_2P^{-1} = PD_2D_1P^{-1} = (PD_2P^{-1})(PD_1P^{-1}) = BA.$$

8. Since the eigenvalues of the matrix $A_{a,b}$ are $\lambda_1 = 1$ with $m_a(1) = 3$ and $\lambda_2 = a$, with $m_a(a) = 1$, the matrix is diagonalizable if and only if $a = 1$ and $b = 0$, or if $a \neq 1$, every b.

Please note. Remember that in general there might be more than one technique to solve the same exercise. If you find a typo, or something that you do not understand, let me know!