Linear algebra and geometry a.y. 2024-2025

Worksheet 11: exercises on chapter 17 from the lecture notes

1. Let $v_1 = (1,0,0,0)$, $v_2 = (1,3,5,0)$, $v_3 = (3,2,-1,1)$, $v_4 = (1,1,0,0)$ be vectors in $\mathbb{R}^4$, and $w_1 = (1,0,1)$, $w_2 = (1,1,0)$, $w_3 = (1,0,0)$ vectors in $\mathbb{R}^3$.

   (a) Check that the set $\mathcal{B} = (v_1, v_2, v_3, v_4)$ is a basis of $\mathbb{R}^4$.

   (b) Check that the set $\mathcal{D} = (w_1, w_2, w_3)$ is a basis of $\mathbb{R}^3$.

   (c) We know that there exists a unique linear map $f : \mathbb{R}^4 \to \mathbb{R}^3$ such that:

   $$\begin{aligned} f(v_1) &= w_1 + w_3 \\ f(v_2) &= -w_1 + w_2 \\ f(v_3) &= w_3 \\ f(v_4) &= 3w_1 + 2w_2 - w_3 \end{aligned}$$

   Find the matrix $M^{\mathcal{B}}_{\mathcal{D}}(f)$ associated to f with respect to the bases $\mathcal{B}$ and $\mathcal{D}$.

   (d) Then find the matrix $M^{\mathcal{C}}_{\mathcal{C}'}(f)$, where $\mathcal{C} = (e_1, e_2, e_3, e_4)$ and $\mathcal{C}' = (e'_1, e'_2, e'_3)$ are the canonical bases of $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively.

2. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear maps defined by

   $$\varphi(x, y) = (y, x + y, x - y).$$

   Find the matrix associated to φ with respect to the bases $\mathcal{B} = (u_1, u_2)$ of $\mathbb{R}^2$ and $\mathcal{D} = (v_1, v_2, v_3)$ of $\mathbb{R}^3$, where:

   $$u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \qquad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

3. The linear map f defined by

   $$f(p(x)) = p'(x) + p(0)$$

   send the space of polynomials $\mathbb{R}[x]_2$ to $\mathbb{R}[x]_1$. Write down the matrix $M^{\mathcal{C}}_{\mathcal{B}}(f)$ with respect to the bases $\mathcal{C} = (1, x, x^2)$ and $\mathcal{B} = (2, 1 - x)$. Find the components of the polynomial $f(x^2 + 1)$ in the basis $\mathcal{B}$.

4. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$g(x, y, z) = (x + y, x + y, z).$$

(a) Write down the matrix $M_{\mathcal{C}}^{\mathcal{C}}(g)$ associated to g with respect to the canonical basis $\mathcal{C}$ of $\mathbb{R}^3$.

(b) Find a basis and compute the dimension of $\mathrm{Ker}(g)$ and $\mathrm{Im}(g)$.

(c) Prove that the set

$$\mathcal{B} = (b_1 = (1, 1, -1),\ b_2 = (1, 1, 0),\ b_3 = (1, -1, 0))$$

is a basis of $\mathbb{R}^3$, then write down the matrix $M_{\mathcal{B}}^{\mathcal{C}}(g)$ associated to g with respect to the canonical basis in the domain and the basis $\mathcal{B}$ in the codomain.

(d) Find the matrices of change of basis from $\mathcal{B}$ to the canonical basis $\mathcal{C}$ and from $\mathcal{C}$ to $\mathcal{B}$.

5. In the vector space $\mathbb{R}[x]_2$, consider the polynomials

$$p_1(x) = x^2 - 2x, \quad p_2(x) = 1 + 2x, \quad p_3(x) = 2 - x^2,$$

$$q_1(x) = -1 + x, \quad q_2(x) = -1 + x - x^2, \quad q_3(x) = 2x + 2x^2.$$

Show that $\mathcal{B} = (p_1, p_2, p_3)$ and $\mathcal{D} = (q_1, q_2, q_3)$ are two bases of $\mathbb{R}[x]_2$. Find the matrix of change of basis from $\mathcal{B}$ to $\mathcal{D}$.

6. Let V and W be two vector spaces over $\mathbb{R}$; V has $\dim_{\mathbb{R}} = 4$ and (v_1, v_2, v_3, v_4), while W has $\dim_{\mathbb{R}} = 3$ and (w_1, w_2, w_3). Find kernel and image of the linear map $\varphi : V \to W$ defined by:

$$\begin{aligned} \varphi(v_1) &= w_1 + w_2, \\ \varphi(v_2) &= w_1 - w_2 + w_3, \\ \varphi(v_3) &= w_2, \\ \varphi(v_4) &= w_1 + w_3. \end{aligned}$$

Solutions.

1. (a) $\det\begin{pmatrix}1&0&0&0\\1&3&5&0\\3&2&-1&1\\1&1&0&0\end{pmatrix}=5\neq 0$, hence the v_is are linearly independent, and since they are 4, they are also generators;

   (b) $\det\begin{pmatrix}1&0&1\\1&1&0\\1&0&0\end{pmatrix}=-1\neq 0$, hence the w_is are linearly independent, and since they are 3, they are also generators;

   (c) $M^{\mathcal{B}}_{\mathcal{D}}(f)=\begin{pmatrix}1&-1&0&3\\0&1&0&2\\1&0&1&-1\end{pmatrix}$;

   (d) $M^{\mathcal{C}}_{\mathcal{C}'}(f)=\begin{pmatrix}2&2&-\frac{8}{5}&-\frac{53}{5}\\0&2&-1&-5\\1&2&-\frac{8}{5}&-\frac{43}{5}\end{pmatrix}$.

2. $M^{\mathcal{B}}_{\mathcal{D}}(\varphi)=\begin{pmatrix}-1&-3\\4&2\\-1&2\end{pmatrix}$

3. $M^{\mathcal{C}}_{\mathcal{B}}(f)=\begin{pmatrix}1/2&1/2&1\\0&0&-2\end{pmatrix}$;
   $[f(x^2+1)]_{\mathcal{B}}=(3/2,-2)$, indeed $f(x^2+1)=2x+1=(3/2)(2)-2(1-x)$.

4. (a) $M^{\mathcal{C}}_{\mathcal{C}}(g)=\begin{pmatrix}1&1&0\\1&1&0\\0&0&1\end{pmatrix}$;

   (b) $\text{Ker}(g)=\mathcal{L}((-1,1,0))$, so $\dim(\text{Ker}(g))=1$; $\text{Im}(g)=\mathcal{L}((1,1,0),(0,0,1))$, so $\dim(\text{Im}(g))=2$;

   (c) $\det\begin{pmatrix}1&1&-1\\1&1&0\\1&-1&0\end{pmatrix}=2\neq 0$: the b_is are 3 linearly independent vectors in $\mathbb{R}^3$, hence they are a basis;

   $M^{\mathcal{C}}_{\mathcal{B}}(g)=\begin{pmatrix}0&0&-1\\1&1&1\\0&0&0\end{pmatrix}$;

   (d) $M^{\mathcal{B}}_{\mathcal{C}}(id_{\mathbb{R}^3})=\begin{pmatrix}1&1&1\\1&1&-1\\-1&0&0\end{pmatrix}$, $M^{\mathcal{C}}_{\mathcal{B}}(id_{\mathbb{R}^3})=(M^{\mathcal{B}}_{\mathcal{C}}(id_{\mathbb{R}^3}))^{-1}=\begin{pmatrix}0&0&-1\\1/2&1/2&1\\1/2&-1/2&0\end{pmatrix}$.

5. Since
$$\det\begin{pmatrix}0 & -2 & 1\\ 1 & 2 & 0\\ 2 & 0 & -1\end{pmatrix} = -6 \neq 0, \quad \det\begin{pmatrix}-1 & 1 & 0\\ -1 & 1 & -1\\ 0 & 2 & 2\end{pmatrix} = -6 \neq 0,$$
both the polynomials p_is and the q_js are 4 linearly independent vectors in $\mathbb{R}[x]_3$, hence they are two bases;
$$M^{\mathcal{B}}_{\mathcal{D}}(id_{\mathbb{R}[x]_3}) = \begin{pmatrix} 3 & -4 & -5\\ -3 & 3 & 3\\ -1 & 3/2 & 1\end{pmatrix}.$$

6. $\mathrm{Ker}(\varphi) = \mathcal{L}(v_2 + v_3 - v_4)$ has dim $= 1$; $\mathrm{Im}(\varphi) = W$, that is, φ is surjective.

Please note. Remember that in general there might be more than one technique to solve the same exercise. If you find a typo, or something that you do not understand, let me know!