Linear algebra and geometry a.y. 2024-2025

Worksheet 10: exercises on chapter 16 from the lecture notes

1. Verify that the following are linear maps $\mathbb{R}^2 \to \mathbb{R}^2$.
   (a) $f(x, y) = (y, x)$
   (b) $f(x, y) = (-y, x)$
   (c) $f(x, y) = (2x, 2y)$
   (d) $f(x, y) = (0, y)$

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined by
$$f(x, y, z) = (x + y, x + y, z).$$
   (a) Write down the associated matrix $A = M(f)$.
   (b) Determine $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ (finding a basis and computing dimension).

3. Let $g : \mathbb{R}[x]_3 \to \mathbb{R}^{2,2}$ be the linear map sending a degree ≤ 3 polynomial of the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ to the following 2×2 matrix:
$$g(p(x)) = \begin{pmatrix} a_3 & 2a_2 \\ 3a_1 & 4a_0 \end{pmatrix}.$$
   Prove that the map g is both surjective and injective, hence it is an isomorphism.

4. Let $h : \mathbb{R}[x]_3 \to \mathbb{R}^{2,2}$ be the linear map sending a degree ≤ 3 polynomial of the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ to the following 2×2 matrix:
$$h(p(x)) = \begin{pmatrix} a_0 + a_1 & a_0 + a_2 \\ 0 & a_0 + a_3 \end{pmatrix}.$$
   Prove that the map h is neither surjective nor injective. (In particular, not all maps between vector spaces of the same dimension are isomorphisms!)

5. Given the matrix
$$A = \begin{pmatrix} -1 & 0 & 2 & 1 & 0 \\ -6 & 5 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3,5}$$
   Compute the dimension and find a basis of the subspaces $\mathrm{Ker}(\mu_A)$ and $\mathrm{Im}(\mu_A)$.

6. Which pairs among the following vector spaces are isomorphic pairs?
$$\mathbb{R}^7, \quad \mathbb{R}^{12}, \quad \mathbb{R}^{3,3}, \quad \mathbb{R}^{3,4}, \quad \mathbb{R}^{4,3}, \quad \mathbb{R}[x]_6, \quad \mathbb{R}[x]_8, \quad \mathbb{R}[x]_{11}$$

Solutions.

1. Here is the solution for the first one, the other ones are similar. Let (x_1, y_1) and (x_2, y_2) be two elements in $\mathbb{R}^2$ and let $\lambda \in \mathbb{R}$; we show that f respects both the addition and the scalar multiplication:

$$\begin{aligned} f((x_1, y_1) + (x_2, y_2)) &= f(x_1 + x_2, y_1 + y_2) = (y_1 + y_2, x_1 + x_2) = (y_1, x_1) + (y_2, x_2) \\ &= f(x_1, y_1) + f(x_2, y_2) \quad \checkmark \end{aligned}$$

$$f(\lambda(x_1, y_1)) = f(\lambda x_1, \lambda y_1) = (\lambda y_1, \lambda x_1) = \lambda(y_1, x_1) = \lambda f(x_1, y_1) \quad \checkmark$$

2. (a) $A = M(f) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

   (b) $\mathrm{Ker}(f) = \mathcal{L}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, hence $\dim(\mathrm{Ker}(f)) = 1$, while $\mathrm{Im}(f) = \mathcal{L}\Big(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\Big)$, hence $\dim(\mathrm{Im}(f)) = 2$

3. g is injective, because

$$\begin{aligned} \mathrm{Ker}(g) &= \{p(x) \in \mathbb{R}[x]_3 \mid g(p(x)) = 0_{\mathbb{R}^{2,2}}\} \\ &= \{p(x) \in \mathbb{R}[x]_3 \mid g(p(x)) = \begin{pmatrix} a_3 & 2a_2 \\ 3a_1 & 4a_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\} \\ &= \{a_0 + a_1x + a_2x^2 + a_3x^3 \in \mathbb{R}[x]_3 \mid a_3 = 0, 2a_2 = 0, 3a_1 = 0, 4a_0 = 0\} \\ &= \{p(x) = 0\}. \end{aligned}$$

   Moreover, g is surjective: it is immediate to check that any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the image through g of the polynomial $\frac{d}{4} + \frac{c}{3}x + \frac{b}{2}x^2 + ax^3$.

4. We compute

$$\begin{aligned} \mathrm{Ker}(h) &= \{p(x) \in \mathbb{R}[x]_3 \mid h(p(x)) = 0_{\mathbb{R}^{2,2}}\} \\ &= \{p(x) \in \mathbb{R}[x]_3 \mid h(p(x)) = \begin{pmatrix} a_0 + a_1 & a_0 + a_2 \\ 0 & a_0 + a_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\} \\ &= \{a_0 + a_1x + a_2x^2 + a_3x^3 \in \mathbb{R}[x]_3 \mid \begin{cases} a_0 + a_1 = 0 \\ a_0 + a_2 = 0 \\ a_0 + a_3 = 0 \end{cases} \} \\ &= \{a_0 + a_1x + a_2x^2 + a_3x^3 \in \mathbb{R}[x]_3 \mid a_0 = -a_1 = -a_2 = -a_3\}, \end{aligned}$$

   so $\mathrm{Ker}(h) \neq \{0\}$, h is not injective.

   The map h is not surjective, either: just notice that any matrix whose entry $(2, 1)$ is nonzero will never belong to the image of h.

5. $\mathrm{Im}(\mu_A) = \mathcal{L}\left(\begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right)$ is of dimension 3;

   $\mathrm{Ker}(\mu_A) = \mathcal{L}\left(\begin{pmatrix} 1 \\ 6/5 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/5 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$ is of dimension 2.

6. $(\mathbb{R}^7, \mathbb{R}[x]_6)$,
   $(\mathbb{R}^9, \mathbb{R}^{3,3})$,
   $(\mathbb{R}^{12}, \mathbb{R}^{3,4})$, $(\mathbb{R}^{12}, \mathbb{R}[x]_{11})$, $(\mathbb{R}^{3,4}, \mathbb{R}[x]_{11})$, $(\mathbb{R}^{12}, \mathbb{R}^{4,3})$, $(\mathbb{R}^{4,3}, \mathbb{R}^{3,4})$, $(\mathbb{R}^{4,3}, \mathbb{R}[x]_{11})$.

Please note. Remember that in general there might be more than one technique to solve the same exercise. If you find a typo, or something that you do not understand, let me know!