Linear algebra and geometry a.y. 2023-2024

Mixed quizzes on inner products, orthogonal diagonalization, quadratic forms, conics

1. Let $q(x, y) = x^2 - 8xy - y^2$ be a quadratic form.

   Which of the following statements is true?

   (a) q is positive definite.

   (b) The matrix associated to q is $\begin{pmatrix} 1 & -4 \\ -4 & -1 \end{pmatrix}$.

   (c) q is negative definite.

   (d) The matrix associated to q is $\begin{pmatrix} -1 & -4 \\ -4 & 1 \end{pmatrix}$.

2. Given the polynomial

$$p(t) = (t^2 + t + 1)(t^2 - 2t + 1)\,,$$

   which of the following statements is true?

   (a) There exists a symmetric positive definite matrix $A \in \mathbb{R}^{4,4}$ having $p(t)$ as characteristic polynomial.

   (b) If a matrix A has $p(t)$ as characteristic polynomial, then A has a 3-dimensional eigenspace.

   (c) None of the other statements is true.

   (d) There is no real symmetric matrix whose characteristic polynomial is $p(t)$.

3. Consider the quadratic form

$$q(x, y, z) = (x, y, z) \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

   Which of the following statements is true?

   (a) The matrix associated to $q(x, y, z)$ has nonzero determinant.

   (b) There exists $(0, 0, 0) \neq (a, b, c) \in \mathbb{R}^3$ such that $q(a, b, c) = 0$.

   (c) $q(x, y, z)$ is positive definite.

   (d) $Q(x, y) = q(x, y, y)$ is positive definite.

4. Let A be a real symmetric 5×5 matrix with zero trace and zero determinant.

   Which of the following statements is true?

   (a) 0 is an eigenvalue of A.
   (b) A is (positive or negative) definite.
   (c) A is (positive or negative) semidefinite.
   (d) None of the other statements is true.

5. Let A be a real 3×3 symmetric matrix having an eigenvalue equal to 1. If the eigenspace $E_A(1)$ relative to the eigenvalue 1 is the set of vectors (x, y, z) such that $x + y - z = 0$, which of the following statements is true?

   (a) A is not diagonalizable.
   (b) $(1, 1, -1)$ is an eigenvector of A relative to the eigenvalue 1.
   (c) $(1, 1, -1)$ is not an eigenvector of A.
   (d) $(1, 1, -1)$ is an eigenvector of A relative to an eigenvalue different from 1.

6. Consider the quadratic form with real coefficients

$$q(x, y, z) = 10x^2 + 4y^2 + 4yz + z^2.$$

   Which of the following statements is true?

   (a) There exists $(a, b, c) \in \mathbb{R}^3$ such that $q(a, b, c) > 0$.
   (b) None of the other statements is true.
   (c) There is no $(a, b, c) \in \mathbb{R}^3$ with $(a, b, c) \neq (0, 0, 0)$ such that $q(a, b, c) = 0$.
   (d) For all $(a, b, c) \in \mathbb{R}^3$, $q(a, b, c) \leq 0$.

7. Given the quadratic form

$$f(x, y) = (x, y)A \begin{pmatrix} x \\ y \end{pmatrix} = x^2 - 3xy + 8y^2,$$

   which of the following statements is true?

   (a) The determinant of the matrix A is a negative number.
   (b) There exists $(a, b) \in \mathbb{R}^2$ such that $f(a, b) < 0$.
   (c) If $xy \neq 0$, then $f(x, y) > 0$.
   (d) The determinant of the matrix A cannot be computed.

8. Consider the quadratic form

$$q(x,y,z) = (x,y,z)B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -x^2 + 2y^2 + 2xz + z^2.$$

Which of the following statements is true?

(a) The matrix B admits both positive and negative eigenvalues.
(b) $q(x,y,x+y) = 0$.
(c) $q(y+z,y,z) = 0$.
(d) The matrix B has rank 2.

9. Let h be a real parameter, and consider in the Euclidean plane the family of conics described by the equation

$$x^2 + 12xy + 11y^2 + h - 1 = 0.$$

Which of the following statements is true?

(a) $\mathcal{C}_h$ is a parabola, for all values of h.
(b) $\mathcal{C}_h$ is an ellipse, for all values of h.
(c) When $h = 1$, the conic is degenerate.
(d) None of the other statements is true.

10. In the Euclidean plane with a fixed coordinate system Oxy, consider the conic $\mathcal{C}$ of equation $2x^2 + 5y^2 - 2xy = 0$.

Which of the following statements is true?

(a) $\mathcal{C}$ is a double line.
(b) $\mathcal{C}$ is a parabola.
(c) $\mathcal{C}$ is non degenerate.
(d) $\mathcal{C}$ is the union of two lines.

11. Find the negative definite quadratic form.

(a) $x^2 + y^2 - 100xy$.
(b) $y^2 - x^2 - 100xy$.
(c) $2xy - 3x^2 - 2y^2$.
(d) $-2x^2 - y^2 - 6xy$.

12. Consider the family of conics described by the equation

$$x^2 + kxy + y^2 + kx - 1 = 0,$$

where $k \in \mathbb{R}$ is a real parameter.

Find the true statement.

(a) There exists precisely two values of k such that the equation represents a hyperbola.
(b) There is no value of k such that the equation represents a degenerate conic.
(c) There is no value of k such that the equation represents a parabola.
(d) None of the other statements is true.

13. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, and let $p_A(t)$ be its characteristic polynomial.

Find the true statement.

(a) $p_A(t) = t^4 + t^3 + t^2 + t + 1$.
(b) $p_A(t) = t^4 + t^3 + t$.
(c) $p_A(t) = t^4 - 4t^3$.
(d) $p_A(t) = -t^4 + t^3 + t^2 + t$.

14. Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 10 \end{pmatrix} \in \mathbb{R}^{3,3}.$$

Which of the following statements is true?

(a) A is positive semidefinite.
(b) A does not have positive eigenvalues.
(c) A does not have a 0 eigenvalue.
(d) None of the other statements is true.