Linear algebra and geometry a.y. 2023-2024

Mixed quizzes on linear maps, associated matrices, diagonalization

1. Let A, B be two $n \times n$ matrices such that $0 < \mathrm{rk}(A) < \mathrm{rk}(B) = n$.

   Find the true statement.

   (a) $\mathrm{rk}(AB) = n\mathrm{rk}(B)$
   (b) AB is invertible
   (c) $\det(AB) = \det(A)$
   (d) $\lambda = 0$ is not an eigenvalue of AB

2. Let $A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$.

   Find the true statement.

   (a) $A^2 = A^3$
   (b) $\lambda = 4$ is not an eigenvalue of A^2
   (c) $(1, 1)$ is an eigenvector of A
   (d) $\det(A^3) = 8$

3. Let $A \in \mathbb{R}^{4,5}$ and $B \in \mathbb{R}^{5,4}$ be two matrices, and consider the endomorphism $f : \mathbb{R}^5 \to \mathbb{R}^5$ associated to the product BA.

   Find the true statement.

   (a) f is not an isomorphism
   (b) f is injective but not surjective
   (c) $\det(BA) \neq 0$
   (d) None of the other statements is true.

4. Consider the linear map $f(x, y, z) = (x + y + z, 0, 0)$.

   Find the true statement.

   (a) f is surjective
   (b) f is not injective
   (c) $(1, 1, 1)$ is sent by f into a multiple of itself.
   (d) $\dim \mathrm{Ker}(f) = 3$

5. Consider the linear map $f : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $f(x, y, z, t) = (x + y, z + t)$.

   Find the true statement.

   (a) f is injective
   (b) f is surjective
   (c) $f^{-1}(2, 2) = \{(1, 1, 1, 1)\}$
   (d) $(1, 1, 1, 1) \in \mathrm{Ker}(f)$

6. Let $\mathbb{R}[x]_3$ be the vector space of polynomials in the variable x with real coefficients and degree ≤ 3 and consider the linear map $f : \mathbb{R}[x]_3 \longrightarrow \mathbb{R}[x]_3$ such that

$$f(p(x)) = p(0)x^2.$$

   Find the true statement.

   (a) f is injective
   (b) f is an isomorphism
   (c) $\dim(\mathrm{Ker}(f)) = 2$
   (d) $\dim(\mathrm{Im}(f)) = 1$

7. Let A be an $n \times n$ matrix having eigenvalue $\lambda = 1$ with algebraic multiplicity $m_a(1) = n$.

   Find the true statement.

   (a) A power of $A - I_n$ is the zero matrix.
   (b) A is not invertible
   (c) A power of $A + I_n$ is the zero matrix
   (d) $A - I_n$ is invertible

8. Consider the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(1, 2) = (2, 1)$ and $f(2, 1) = (1, 2)$.

   Find the true statement.

   (a) f is not surjective
   (b) $f^{-1}(3, 3) = \{(1, 1)\}$
   (c) f is injective
   (d) $\{(0, -1), (1, 0)\}$ is a set of generators of $\mathrm{Ker}(f)$

9. Let $\mathbb{R}[x]_2$ be the vector space of polynomials in x, with real coefficients and degree ≤ 2. Define the endomorphism $f\colon \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ as:

$$f(ax^2 + bx + c) = cx^2 - b,$$

and let $M_{\mathcal{B}}^{\mathcal{B}}(f)$ be the matrix associated to f with respect to the basis $\mathcal{B} = (x^2, x, 1)$ (both in the domain and codomain).

Which of the following statements is true?

(a) $M_{\mathcal{B}}^{\mathcal{B}}(f) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(b) $M_{\mathcal{B}}^{\mathcal{B}}(f) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$

(c) $M_{\mathcal{B}}^{\mathcal{B}}(f) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$

(d) $M_{\mathcal{B}}^{\mathcal{B}}(f) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

10. Let f be an endomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(1,1,0) = f(1,0,1) = (0,1,1)$.

Which of the following statements is true?

(a) None of the other statements is true.

(b) f is surjective.

(c) $\dim(\operatorname{Ker} f) \geq 1$.

(d) f is injective.

11. Let $\vec{w} \in \mathbb{R}^3$ be a unit vector. Consider the endomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$f(\vec{v}) = \vec{v} - (\vec{v} \cdot \vec{w})\, \vec{w}\,.$$

Which of the following statements is true?

(a) f is neither injective nor surjective.

(b) f is surjective.

(c) f is injective.

(d) $\vec{w}$ does not belong to $\operatorname{Ker}(f)$.

12. Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be an endomorphism with characteristic polynomial

$$p_f(t) = (t^2 - 7)(t^2 - 4t).$$

Which of the following statements is true?

(a) $\dim(\mathrm{Ker}(f)) = 2$
(b) $\dim(\mathrm{Im}(f)) = 4$
(c) f is injective
(d) $\dim(\mathrm{Im}(f)) = 3$

13. Given a linear map $f : \mathbb{R}^4 \to \mathbb{R}^2$, which of the following statements is surely false?

(a) $\dim \mathrm{Ker}(f) = 2$.
(b) f is surjective.
(c) $\dim \mathrm{Ker}(f) = 1$.
(d) The matrix associated to f has rank 2.

14. Consider the endomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$f(x, y, z) = (x + z, 2y, x + z).$$

Which of the following statements is true?

(a) f is an isomorphism.
(b) f is diagonalizable.
(c) f is not simple.
(d) All eigenspaces of f have dimension 1.

15. Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

which of the following statements is true?

(a) The characteristic polynomial of A is $-t^3 + 6t^2$.
(b) The characteristic polynomial of A is $-t^3 + 6t$.
(c) The characteristic polynomial of A is $-t^3 + 6$.
(d) The characteristic polynomial of A is $t^3 + 6t^2$.

16. Given the endomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ with characteristic polynomial $-(t-1)t^2$, which of the following statements is true?

   (a) If the image of f has dimension 1, then f is diagonalizable.
   (b) f has 3 linearly independent eigenvectors.
   (c) 4 is an eigenvalue of f.
   (d) If $\dim(\mathrm{Ker}(f)) = 1$, then f is diagonalizable.

17. Let $V = \{(x, y, z) \in \mathbb{R}^3 : x + y - z = 0\}$ and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be an endomorphism such that $V = \mathrm{Ker}(f)$ and 1 is an eigenvalue of f.

   Which of the following statements is true?

   (a) f has 3 distinct eigenvalues.
   (b) $\dim(\mathrm{Im}(f)) = 2$.
   (c) f is diagonalizable.
   (d) The characteristic polynomial of f could be $-t(t-1)^2$.

18. Let g be an endomorphism of $\mathbb{R}^3$ and let

$$M_{\mathcal{C}}^{\mathcal{C}}(g) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 4 & 1 \\ 5 & 0 & 3 \end{pmatrix}$$

   be the matrix associated to g with respect to the canonical basis $\mathcal{C} = (e_1, e_2, e_3)$ of $\mathbb{R}^3$.

   Which of the following statements is true?

   (a) e_1 is an eigenvector of g.
   (b) e_3 is an eigenvector of g.
   (c) None of the other statements is true.
   (d) e_2 is an eigenvector of g.