Linear algebra and geometry a.y. 2023-2024
Mixed quizzes on vector spaces

1. In $\mathbb{R}^4$, define the following vector subspaces:

$$W_1 = \{(x, y, z, t) \in \mathbb{R}^4 \,|\, x + y + z + t = 0\}$$
$$W_2 = \{(x, y, z, t) \in \mathbb{R}^4 \,|\, \begin{cases} x - y = 0 \\ z + t = 0 \end{cases} \}.$$

Which of the following statements is true?

(a) $\dim(W_1 + W_2) = 4$.
(b) $\dim(W_1 + W_2) = 3$.
(c) $\dim(W_1 + W_2) = 2$.
(d) $\dim(W_1 + W_2) = 1$.

2. Let $\mathcal{C} = (e_1, e_2, e_3, e_4)$ be the canonical basis of $\mathbb{R}^4$, and define the subspace

$$V = \mathcal{L}(e_2 + e_3,\ -e_1 + e_3,\ -e_1 - e_2).$$

The dimension of V is

(a) 3
(b) 1
(c) 2
(d) 4

3. Let $\mathcal{C} = (e_1, e_2, e_3)$ be the canonical basis of $\mathbb{R}^3$, and consider the vectors

$$v_1 = e_1 + e_2 + e_3, \qquad v_2 = -3e_2, \qquad v_3 = 2e_1 + 5e_2 + 2e_3, \qquad v_4 = e_3 - e_1.$$

Find the true statement.

(a) The set $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.
(b) The set $\{v_1, v_2, v_3, v_4\}$ generates $\mathbb{R}^3$.
(c) The set $\{v_1, v_3\}$ generates $\mathbb{R}^3$.
(d) The set $\{v_1, v_2, v_3\}$ is linearly independent in $\mathbb{R}^3$.

4. Let $\mathbb{R}[x]_3$ be the vector space of polynomials in the variable x, with real coefficients and degree ≤ 3, and consider the vectors

$$v_1 = 1 + x, \quad v_2 = 1 + x^2, \quad v_3 = x + x^2, \quad v_4 = x^3.$$

Find the true statement.

(a) The set $\{v_1, v_2\}$ generates $\mathbb{R}[x]_3$.
(b) The set $\{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}[x]_3$.
(c) The set $\{v_1 + v_2, v_3 + v_4\}$ is a basis of $\mathbb{R}[x]_3$.
(d) The set $\{v_1, v_2, v_3\}$ is linearly dependent.

5. Define the following subsets of $\mathbb{R}^{2,2}$:

$$V = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbb{R} \right\} \quad \text{and} \quad W = \left\{ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \mid c \in \mathbb{R} \right\}.$$

Find the true statement.

(a) The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ form a basis of W
(b) W is not a vector subspace of V
(c) $\dim(V) = \dim(W) = 3$
(d) $\dim(V \cap W) = 1$

6. Let $\mathbb{R}^{2,2}$ denote the vector space of 2×2 matrices with real entries. Which of the following subsets of $\mathbb{R}^{2,2}$ is a vector subspace?

(a) The set of matrices in $\mathbb{R}^{2,2}$ with determinant equal to 0.
(b) The set of matrices in $\mathbb{R}^{2,2}$ such that the sum of all entries is 0.
(c) The set of matrices in $\mathbb{R}^{2,2}$ such that the product of the entries in the first column is 0.
(d) The set of matrices in $\mathbb{R}^{2,2}$ such that the product of the entries on the main diagonal is 0.

7. Let $U \subseteq \mathbb{R}^4$ be a vector subspace of dimension 2.

Which of the following statements is true?

(a) There exists a subspace V of dimension 3 such that $\dim(U \cap V) = 2$.
(b) There exists a subspace V of dimension 2 that does not intersect U.
(c) For all subspaces V of dimension 3, one has $\dim(U \cap V) = 1$.
(d) None of the other statements is true.

8. Given the basis $\mathcal{B} = \big((1,2,3),(1,0,1),(0,1,2)\big)$ of $\mathbb{R}^3$ and the vector $v = (3,1,3)$, which of the following statements is true?

   (a) The vector v is not a linear combination of the elements of $\mathcal{B}$.
   (b) The components of v in the basis $\mathcal{B}$ are $[v]_{\mathcal{B}} = (1,-1,1)$.
   (c) The components of v in the basis $\mathcal{B}$ are $[v]_{\mathcal{B}} = (3,1,3)$.
   (d) The components of v in the basis $\mathcal{B}$ are $[v]_{\mathcal{B}} = (1,2,-1)$.

9. In $\mathbb{R}^4$, define the following vector subspaces:

$$V = \{\ (x,y,z,w) \in \mathbb{R}^4 \mid x+y+z-w = x+3y = 0\ \},$$

$$W = \{\ (x,y,z,w) \in \mathbb{R}^4 \mid 2y-z+w = x = 0\ \}.$$

   Which of the following statements is true?

   (a) The vector $(0,0,1,0)$ is a basis of $V \cap W$.
   (b) $V + W = \mathbb{R}^4$.
   (c) $\dim(V \cap W) = 2$.
   (d) $V \cup W$ is contained in a vector subspace in $\mathbb{R}^4$ of dimension 3.

10. Let $h \in \mathbb{R}$ and consider the elements of $\mathbb{R}[x]_2$:

$$p_1(x) = x^2 + hx, \qquad p_2(x) = x + h, \qquad p_3(x) = hx^2 + 1.$$

   Which of the following statements is true?

   (a) When $h = 0$, (p_1, p_2, p_3) is not a basis of $\mathbb{R}[x]_2$.
   (b) $\{p_1, p_2, p_3\}$ is a set of linearly dependent polynomials for all $h \in \mathbb{R}$.
   (c) There exists a value of h such that the vector subspace generated by $\{p_1, p_2, p_3\}$ has dimension 2.
   (d) When $h = -1$, $\{p_1, p_2, p_3\}$ is a set of linearly independent polynomials.

11. Which of the following subsets in $\mathbb{R}[x]_3$ is a vector subspace?

   (a) The set $\big\{p(x) = a_0 + a_1x + a_2x^2 + a_3 \in \mathbb{R}[x]_3 \,|\, a_0a_2 = 0\big\}$.
   (b) The set $\big\{p(x) = a_0 + a_1x + a_2x^2 + a_3 \in \mathbb{R}[x]_3 \,|\, a_0 + a_1 + a_2 + a_3 = 0\big\}$.
   (c) The set $\big\{p(x) = a_0 + a_1x + a_2x^2 + a_3 \in \mathbb{R}[x]_3 \,|\, a_0a_3 = 0\big\}$.
   (d) The set $\big\{p(x) = a_0 + a_1x + a_2x^2 + a_3 \in \mathbb{R}[x]_3 \,|\, a_0a_3 - a_1a_2 = 0\big\}$.

12. In the vector space $\mathbb{R}^{2,2}$ of 2×2 matrices with real entries, consider the elements

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Which of the following statements is true?

(a) A_1, A_2, A_3, A_4 are linearly dependent.
(b) (A_1, A_2, A_3, A_4) is a basis of $\mathbb{R}^{2,2}$.
(c) $\{A_1, A_2\}$ and $\{A_3, A_4\}$ generate the same vector subspace of $\mathbb{R}^{2,2}$.
(d) A_1, A_2, A_3 generate a vector subspace of dimension 2.

13. Let $h \in \mathbb{R}$ and let U be the vector subspace of $\mathbb{R}^3$ generated by the vector $(h, 0, h)$. Let $V = \{(x, y, z) \in \mathbb{R}^3 : x + 3y + z = 0\}$.

Which of the following statements is true?

(a) $\dim(U \cap V) = 1$ for all values of h.
(b) If $h = 0$ then $U \cap V = \emptyset$.
(c) $\dim(U \cap V) = 0$ for all values of h.
(d) $\dim(U) = 1$ for all values of h.

14. Let $h \in \mathbb{R}$ and let U be the vector subspace of $\mathbb{R}^3$ generated by the vectors $(h, 0, 1)$ and $(1, 0, 1)$. Let $V = \{(x, y, z) \in \mathbb{R}^3 : x + 3y = z = 0\}$.

Which of the following statements is true?

(a) There exists a value of h such that $\dim(U \cap V) = 1$.
(b) If $h = 1$ then $U \cap V = \emptyset$.
(c) $\dim(U) = 2$ for all values of h.
(d) $\dim(U + V) = 2$ if and only if $h = 1$.

15. Given the geometric vectors $\vec{v_1} = \vec{\imath} + \vec{\jmath}$, $\vec{v_2} = \vec{\imath} + 2\vec{\jmath}$, and $\vec{v_3} = 2\vec{\imath} + \vec{\jmath}$, find the correct statement.

(a) There exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 \vec{v_1} + \lambda_2 \vec{v_2} = \vec{v_3}$.
(b) $\mathcal{L}(\vec{v_1}, \vec{v_2}, \vec{v_3}) = \mathbb{R}^3$.
(c) $\lambda \vec{v_1} + \vec{v_2} = \vec{v_1}$ for some value $\lambda \in \mathbb{R}$.
(d) The vectors $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ are linearly independent.