Linear algebra and geometry a.y. 2023-2024

Mixed quizzes on matrices and linear systems

1. Consider the linear system $AX = B$ where

$$A = \begin{pmatrix} -1 & 3 & k \\ -1 & 3 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad \text{e} \qquad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

and $k \in \mathbb{R}$ is a parameter.

Find the correct statement.

(a) The system has a unique solution for a unique value of $k \in \mathbb{R}$.
(b) The system has a unique solution for all values of $k \in \mathbb{R}$.
(c) None of the other statements is correct.
(d) There are no values of k such that the system has ∞^1 solutions.

2. Consider the linear system

$$\begin{cases} 2x + y - z = 1 \\ x + y - z = 0 \end{cases}$$

Find the correct statement.

(a) The system has ∞^2 solutions.
(b) $(1, 0, 1)$ is a solution.
(c) $(-1, 1, 0)$ is a solution.
(d) If $z = 0$, the system has no solutions.

3. Which of the following statements is true?

(a) If $A \in \mathbb{R}^{m,n}$ is such that the system $AX = B$ is compatible for all $B \in \mathbb{R}^{m,1}$, then $m \leq n$.
(b) For any choice of positive integers m, n and $A \in \mathbb{R}^{m,n}$ the matrix equation $AX = {}^tAX$ has solution(s).
(c) There exists a nonzero $B \in \mathbb{R}^{m,1}$ such that the system $AX = B$ is compatible for all $A \in \mathbb{R}^{m,n}$.
(d) None of the other statements is true.

4. Let $x \in \mathbb{R}$ and consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & x \end{pmatrix}$$

Find the true statement.

(a) A is invertible for all values of x.
(b) $\text{rk}(A) = 2$ for all values of x.
(c) $\det(A) = 0$ for finitely many values of x.
(d) $\text{rk}(A) = 3$ for finitely many values of x.

5. Let $A \in \mathbb{R}^{3,5}$ such that $\text{rk}(A) = 3$. Which of the following statements is true?

(a) The system $AX = 2A$ has infinitely many solutions.
(b) The system ${}^tAX = 3{}^tA$ has infinitely many solutions.
(c) The system $AX = 4A$ has a unique solution.
(d) None of the other statements is true.

6. Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be an upper triangular matrix such that $\det(A) \neq 0$. Which of the following statements is true?

(a) A^{-1} is not necessarily upper triangular.
(b) None of the other statements is true.
(c) A^{-1} is lower triangular.
(d) A^{-1} is upper triangular and the entry (i, i) of A^{-1} equals $1/a_{ii}$.

7. Let $A \in \mathbb{R}^{3,3}$ be a 3×3 matrix, not symmetric. Which of the following statements is true?

(a) $\det(A - {}^tA) \neq 0$.
(b) $\det(A - {}^tA) = 0$.
(c) $\text{rk}(A - {}^tA) = 1$.
(d) $A - {}^tA = 0_{3,3}$.

8. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Which of the following statements is true?

(a) $(3A)^n = 3^n 2^{n-1} A$

(b) $(3A)^n = 3^n A$

(c) $(3A)^n = 3^{n+1} A$

(d) $(3A)^n = 3^{n-1} A$

9. Consider the matrices

$$E = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \in \mathbb{R}^{1,4}, \quad F = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \in \mathbb{R}^{4,1}.$$

Let A be the product $F \cdot E$.

Which of the following statements about the linear system $AX = B$ is true?

(a) The system $AX = B$ with 4 equations in 4 variables has no solutions for any $B \in \mathbb{R}^{4,1}$.

(b) The system $AX = B$ with 4 equations in 4 variables is compatible for all $B \in \mathbb{R}^{4,1}$.

(c) The system $AX = B$ with 4 equations in 4 variables is compatible if and only if $B = \lambda F$, with $\lambda \in \mathbb{R}$.

(d) If $AX = B$ with 4 equations in 4 variables is compatible, then it has a unique solution.

10. Consider the matrices

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Which of the following statements is true?

(a) There exist infinitely many matrices $X \in \mathbb{R}^{3,2}$ such that $AX = B$.

(b) There exists a unique matrix $X \in \mathbb{R}^{3,2}$ such that $AX = B$.

(c) There exist exactly two matrices $X, X' \in \mathbb{R}^{3,2}$ such that $AX = AX' = B$.

(d) There do not exist matrices $X \in \mathbb{R}^{3,2}$ such that $AX = B$.

11. Let $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and let $B \in \mathbb{R}^{2,2}$ be an invertible matrix.

Find the true statement.

(a) The matrices BA and $\begin{pmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 8 \end{pmatrix}$ are row equivalent.

(b) The matrices BA and $\begin{pmatrix} 0 & 0 & 8 & 0 \\ 0 & 0 & 8 & 0 \end{pmatrix}$ are row equivalent.

(c) The matrices BA and $\begin{pmatrix} 0 & 8 & 0 & 0 \\ 0 & 8 & 0 & 0 \end{pmatrix}$ are row equivalent.

(d) The matrices BA and $\begin{pmatrix} 8 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 \end{pmatrix}$ are row equivalent.

12. Let A_α be the family of matrices:

$$A_\alpha = \begin{pmatrix} 2 & 2 & 2 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix},$$

with $\alpha \in \mathbb{R}$ a real parameter.

Find the correct statement.

(a) $\text{rk}(A_\alpha) = 2$ for some values of $\alpha \in \mathbb{R}$.

(b) $\text{rk}(A_\alpha) = 1$ for some values of $\alpha \in \mathbb{R}$.

(c) A_α is symmetric for $\alpha = 2$.

(d) A_α is invertible for all values of $\alpha \in \mathbb{R}$.

13. It is known that the linear system

$$A \begin{pmatrix} x \\ y \end{pmatrix} = B$$

has infinitely many solutions when $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Find the correct statement.

(a) A is invertible.

(b) The rank of A is zero.

(c) The complete matrix $(A\,|B)$ is invertible.

(d) The matrix A and the complete matrix $(A\,|B)$ have the same rank.

14. If the linear system $AX = B$ with n equations in n unknowns has at least two solutions, then

(a) $\det(A) = \det(A^2)$.
(b) $\mathrm{rk}(A) = n - 1$.
(c) $\mathrm{rk}(A) = 0$.
(d) $\det(A) \neq 0$.

15. Consider the following linear system in the unknowns x, y, z:

$$\begin{cases} x + 2y + 3z = 2 \\ x + y + 2z = 1 \\ 2x + y + 3z = 1 \end{cases}$$

Find the true statement.

(a) The system has a unique solution.
(b) The system has ∞^2 solutions.
(c) The general solution of the system has the form $\{x = -t\,,\ y = 1 - t\,,\ z = t\}$, $t \in \mathbb{R}$.
(d) The general solution of the system has the form $\{x = t\,,\ y = 1 - t\,,\ z = t\}$, $t \in \mathbb{R}$.

16. Let $M = \begin{pmatrix} 0 & 2 & 2 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Let $\mathcal{S}$ be the homogeneous linear system whose matrix of coefficients is $N \cdot M$. Which of the following statements is true?

(a) $\mathcal{S}$ admits ∞^4 solutions.
(b) $\mathcal{S}$ admits ∞^1 solutions.
(c) $\mathcal{S}$ admits ∞^2 solutions.
(d) $\mathcal{S}$ admits ∞^3 solutions.

17. Let A be a real $n \times n$ matrix. If the homogeneous linear system $AX = 0$ in n unknowns has a nonzero solution, which of the following statements is true?

(a) $\det(A) = \det(A^2)$.
(b) $\mathrm{rk}(A) = n$.
(c) A is invertible.
(d) None of the other statements is true.

18. For all $k \in \mathbb{R}$, let A_k be the matrix

$$A_k = \begin{pmatrix} k & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$$

Which of the following statements is true?

(a) $\det(A_k) \neq 0$ for all values of k.
(b) $\mathrm{rk}(A_k) = 2$ for two distinct values of k.
(c) None of the other statements is true.
(d) $\mathrm{rk}(A_k) = 2$ for a unique value of k.

19. Let A and B be the following 2×2 matrices:

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find the true statement.

(a) AB is the zero matrix
(b) $\mathrm{rk}(A+B) = 2$
(c) $A - B$ is skew-symmetric
(d) $\mathrm{rk}(BA) = 0$

20. Let $A \in \mathbb{R}^{3,3}$ be a symmetric matrix. Then

(a) $\mathrm{rk}(A) = 3$
(b) $A + A^2$ is also symmetric
(c) $\det(A - {}^tA) \neq 0$
(d) $\mathrm{rk}(A - {}^tA) = 1$

21. Let $A \in \mathbb{R}^{3,3}$ be any matrix. Then

(a) If $(1,1,1)$ and $(1,2,3)$ are solutions of the linear system $AX = 0_{3,1}$, then $(2,3,4)$ is also a solution of $AX = 0_{3,1}$.
(b) If $\det(A) \neq 0$, then the linear system $AX = B$ has infinitely many solutions for some $B \in \mathbb{R}^{3,1}$
(c) If $\det(A) \neq 0$, then the linear system $AX = B$ always has infinitely many solutions
(d) If $(1,1,1)$ and $(1,2,3)$ are solutions of the linear system $AX = B$, $(0,1,2)$ is also a solution of the system $AX = B$, with $B \neq 0_{3,1}$