Linear algebra and geometry a.y. 2023-2024

Worksheet 7: exercises on chapters 16–17 from the lecture notes

(Some of these exercises come from the books by [Schlesinger], [Baldovino-Lanza], [Sernesi], [Leon])

1. Verify that the following are linear maps $\mathbb{R}^2 \to \mathbb{R}^2$.
   (a) $f(x, y) = (y, x)$
   (b) $f(x, y) = (-y, x)$
   (c) $f(x, y) = (2x, 2y)$
   (d) $f(x, y) = (0, y)$

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined by
$$f(x, y, z) = (x + y, x + y, z).$$
   (a) Write down the associated matrix $A = M(f)$.
   (b) Determine $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ (finding a basis and computing dimension).

3. Let $g : \mathbb{R}[x]_3 \to \mathbb{R}^{2,2}$ be the linear map sending a degree ≤ 3 polynomial of the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ to the following 2×2 matrix:
$$g(p(x)) = \begin{pmatrix} a_3 & 2a_2 \\ 3a_1 & 4a_0 \end{pmatrix}.$$
   Prove that the map g is both surjective and injective, hence it is an isomorphism.

4. Let $h : \mathbb{R}[x]_3 \to \mathbb{R}^{2,2}$ be the linear map sending a degree ≤ 3 polynomial of the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ to the following 2×2 matrix:
$$h(p(x)) = \begin{pmatrix} a_0 + a_1 & a_0 + a_2 \\ 0 & a_0 + a_3 \end{pmatrix}.$$
   Prove that the map h is neither surjective nor injective. (In particular, not all maps between vector spaces of the same dimension are isomorphisms!)

5. Given the matrix
$$A = \begin{pmatrix} -1 & 0 & 2 & 1 & 0 \\ -6 & 5 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3,5}$$
   Compute the dimension and find a basis of the subspaces $\mathrm{Ker}(\mu_A)$ and $\mathrm{Im}(\mu_A)$.

6. Which pairs among the following vector spaces are isomorphic pairs?

$$\mathbb{R}^7, \quad \mathbb{R}^{12}, \quad \mathbb{R}^{3,3}, \quad \mathbb{R}^{3,4}, \quad \mathbb{R}^{4,3}, \quad \mathbb{R}[x]_6, \quad \mathbb{R}[x]_8, \quad \mathbb{R}[x]_{11}$$

7. Let $v_1 = (1,0,0,0)$, $v_2 = (1,3,5,0)$, $v_3 = (3,2,-1,1)$, $v_4 = (1,1,0,0)$ be vectors in $\mathbb{R}^4$, and $w_1 = (1,0,1)$, $w_2 = (1,1,0)$, $w_3 = (1,0,0)$ vectors in $\mathbb{R}^3$.

   (a) Check that the set $\mathcal{B} = (v_1, v_2, v_3, v_4)$ is a basis of $\mathbb{R}^4$.

   (b) Check that the set $\mathcal{D} = (w_1, w_2, w_3)$ is a basis of $\mathbb{R}^3$.

   (c) We know that there exists a unique linear map $f : \mathbb{R}^4 \to \mathbb{R}^3$ such that:

$$\begin{aligned} f(v_1) &= w_1 + w_3 \\ f(v_2) &= -w_1 + w_2 \\ f(v_3) &= w_3 \\ f(v_4) &= 3w_1 + 2w_2 - w_3 \end{aligned}$$

   Find the matrix $M^{\mathcal{B}}_{\mathcal{D}}(f)$ associated to f with respect to the bases $\mathcal{B}$ and $\mathcal{D}$.

   (d) Then find the matrix $M^{\mathcal{C}}_{\mathcal{C}'}(f)$, where $\mathcal{C} = (e_1, e_2, e_3, e_4)$ and $\mathcal{C}' = (e'_1, e'_2, e'_3)$ are the canonical bases of $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively.

8. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear maps defined by

$$\varphi(x, y) = (y, x + y, x - y).$$

   Find the matrix associated to φ with respect to the bases $\mathcal{B} = (u_1, u_2)$ of $\mathbb{R}^2$ and $\mathcal{D} = (v_1, v_2, v_3)$ of $\mathbb{R}^3$, where:

$$u_1 = (1,2), \quad u_2 = (3,1), \qquad v_1 = (1,0,0), \quad v_2 = (1,1,0), \quad v_3 = (1,1,1).$$

9. The linear map f defined by

$$f(p(x)) = p'(x) + p(0)$$

   send the space of polynomials $\mathbb{R}[x]_2$ to $\mathbb{R}[x]_1$. Write down the matrix $M^{\mathcal{C}}_{\mathcal{B}}(f)$ with respect to the bases $\mathcal{C} = (1, x, x^2)$ and $\mathcal{B} = (2, 1 - x)$. Find the components of the polynomial $f(x^2 + 1)$ in the basis $\mathcal{B}$.

10. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$g(x, y, z) = (x + y, x + y, z).$$

   (a) Write down the matrix $M^{\mathcal{C}}_{\mathcal{C}}(g)$ associated to g with respect to the canonical basis $\mathcal{C}$ of $\mathbb{R}^3$.

   (b) Find a basis and compute the dimension of $\mathrm{Ker}(g)$ and $\mathrm{Im}(g)$.

(c) Prove that the set

$$\mathcal{B} = (b_1 = (1, 1, -1),\ b_2 = (1, 1, 0),\ b_3 = (1, -1, 0))$$

is a basis of $\mathbb{R}^3$, then write down the matrix $M^{\mathcal{C}}_{\mathcal{B}}(g)$ associated to g with respect to the canonical basis in the domain and the basis $\mathcal{B}$ in the codomain.

(d) Find the matrices of change of basis from $\mathcal{B}$ to the canonical basis $\mathcal{C}$ and from $\mathcal{C}$ to $\mathcal{B}$.

11. In the vector space $\mathbb{R}[x]_2$, consider the polynomials

$$p_1(x) = x^2 - 2x, \quad p_2(x) = 1 + 2x, \quad p_3(x) = 2 - x^2,$$

$$q_1(x) = -1 + x, \quad q_2(x) = -1 + x - x^2, \quad q_3(x) = 2x + 2x^2.$$

Show that $\mathcal{B} = (p_1, p_2, p_3)$ and $\mathcal{D} = (q_1, q_2, q_3)$ are two bases of $\mathbb{R}[x]_2$. Find the matrix of change of basis from $\mathcal{B}$ to $\mathcal{D}$.

12. Let V and W be two vector spaces over $\mathbb{R}$; V has $\dim_{\mathbb{R}} = 4$ and (v_1, v_2, v_3, v_4), while W has $\dim_{\mathbb{R}} = 3$ and (w_1, w_2, w_3). Find kernel and image of the linear map $\varphi : V \to W$ defined by:

$$\begin{aligned}
\varphi(v_1) &= w_1 + w_2, \\
\varphi(v_2) &= w_1 - w_2 + w_3, \\
\varphi(v_3) &= w_2, \\
\varphi(v_4) &= w_1 + w_3.
\end{aligned}$$

Solutions.

1. Here is the solution for the first one, the other ones are similar. Let (x_1, y_1) and (x_2, y_2) be two elements in $\mathbb{R}^2$ and let $\lambda \in \mathbb{R}$; we show that f respects both the addition and the scalar multiplication:

$$\begin{aligned} f((x_1, y_1) + (x_2, y_2)) &= f(x_1 + x_2, y_1 + y_2) = (y_1 + y_2, x_1 + x_2) = (y_1, x_1) + (y_2, x_2) \\ &= f(x_1, y_1) + f(x_2, y_2) \quad \checkmark \end{aligned}$$

$$f(\lambda(x_1, y_1)) = f(\lambda x_1, \lambda y_1) = (\lambda y_1, \lambda x_1) = \lambda(y_1, x_1) = \lambda f(x_1, y_1) \quad \checkmark$$

2. (a) $A = M(f) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

   (b) $\mathrm{Ker}(f) = \mathcal{L}((-1, 1, 0))$, hence $\dim(\mathrm{Ker}(f)) = 1$; $\mathrm{Im}(f) = \mathcal{L}((1, 1, 0), (0, 0, 1))$, hence $\dim(\mathrm{Im}(f)) = 2$

3. g is injective, because

$$\begin{aligned} \mathrm{Ker}(g) &= \{p(x) \in \mathbb{R}[x]_3 \mid g(p(x)) = 0_{\mathbb{R}^{2,2}}\} \\ &= \{p(x) \in \mathbb{R}[x]_3 \mid g(p(x)) = \begin{pmatrix} a_3 & 2a_2 \\ 3a_1 & 4a_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\} \\ &= \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in \mathbb{R}[x]_3 \mid a_3 = 0, 2a_2 = 0, 3a_1 = 0, 4a_0 = 0\} \\ &= \{p(x) = 0\}. \end{aligned}$$

   Moreover, g is surjective: it is immediate to check that any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the image through g of the polynomial $\frac{d}{4} + \frac{c}{3}x + \frac{b}{2}x^2 + ax^3$.

4. We compute

$$\begin{aligned} \mathrm{Ker}(h) &= \{p(x) \in \mathbb{R}[x]_3 \mid h(p(x)) = 0_{\mathbb{R}^{2,2}}\} \\ &= \{p(x) \in \mathbb{R}[x]_3 \mid h(p(x)) = \begin{pmatrix} a_0 + a_1 & a_0 + a_2 \\ 0 & a_0 + a_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\} \\ &= \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in \mathbb{R}[x]_3 \mid \begin{cases} a_0 + a_1 = 0 \\ a_0 + a_2 = 0 \\ a_0 + a_3 = 0 \end{cases} \} \\ &= \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in \mathbb{R}[x]_3 \mid a_0 = -a_1 = -a_2 = -a_3\}, \end{aligned}$$

   so $\mathrm{Ker}(h) \neq \{0\}$, h is not injective.

   The map h is not surjective, either: just notice that any matrix whose entry $(2, 1)$ is nonzero will never belong to the image of h.

5. $\mathrm{Im}(\mu_A) = \mathcal{L}((1,6,0),(0,1,0),(2,2,1))$ is of dimension 3;
$\mathrm{Ker}(\mu_A) = \mathcal{L}((1,6/5,0,1,0),(0,-1/5,0,0,1))$ is of dimension 2.

6. $(\mathbb{R}^7, \mathbb{R}[x]_6)$,
$(\mathbb{R}^9, \mathbb{R}^{3,3})$,
$(\mathbb{R}^{12}, \mathbb{R}^{3,4})$, $(\mathbb{R}^{12}, \mathbb{R}[x]_{11})$, $(\mathbb{R}^{3,4}, \mathbb{R}[x]_{11})$, $(\mathbb{R}^{12}, \mathbb{R}^{4,3})$, $(\mathbb{R}^{4,3}, \mathbb{R}^{3,4})$, $(\mathbb{R}^{4,3}, \mathbb{R}[x]_{11})$.

7. (a) $\det\begin{pmatrix}1&0&0&0\\1&3&5&0\\3&2&-1&1\\1&1&0&0\end{pmatrix} = 5 \neq 0$, hence the v_is are linearly independent, and since they are 4, they are also generators;

(b) $\det\begin{pmatrix}1&0&1\\1&1&0\\1&0&0\end{pmatrix} = -1 \neq 0$, hence the w_is are linearly independent, and since they are 3, they are also generators;

(c) $M^{\mathcal{B}}_{\mathcal{D}}(f) = \begin{pmatrix}1&-1&0&3\\0&1&0&2\\1&0&1&-1\end{pmatrix}$;

(d) $M^{\mathcal{C}}_{\mathcal{C}'}(f) = \begin{pmatrix}2&2&-\frac{8}{5}&-\frac{53}{5}\\0&2&-1&-5\\1&2&-\frac{8}{5}&-\frac{43}{5}\end{pmatrix}$.

8. $M^{\mathcal{B}}_{\mathcal{D}}(\varphi) = \begin{pmatrix}-1&-3\\4&2\\-1&2\end{pmatrix}$

9. $M^{\mathcal{C}}_{\mathcal{B}}(f) = \begin{pmatrix}1/2&1/2&1\\0&0&-2\end{pmatrix}$;
$[f(x^2+1)]_{\mathcal{B}} = (3/2,-2)$, indeed $f(x^2+1) = 2x+1 = (3/2)(2) - 2(1-x)$.

10. (a) $M^{\mathcal{C}}_{\mathcal{C}}(g) = \begin{pmatrix}1&1&0\\1&1&0\\0&0&1\end{pmatrix}$;

(b) $\mathrm{Ker}(g) = \mathcal{L}((-1,1,0))$, so $\dim(\mathrm{Ker}(g)) = 1$; $\mathrm{Im}(g) = \mathcal{L}((1,1,0),(0,0,1))$, so $\dim(\mathrm{Im}(g)) = 2$;

(c) $\det\begin{pmatrix}1&1&-1\\1&1&0\\1&-1&0\end{pmatrix} = 2 \neq 0$: the b_is are 3 linearly independent vectors in $\mathbb{R}^3$, hence they are a basis;

$M^{\mathcal{C}}_{\mathcal{B}}(g) = \begin{pmatrix}0&0&-1\\1&1&1\\0&0&0\end{pmatrix}$;

(d) $M^{\mathcal{B}}_{\mathcal{C}}(id_{\mathbb{R}^3}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$, $M^{\mathcal{C}}_{\mathcal{B}}(id_{\mathbb{R}^3}) = (M^{\mathcal{B}}_{\mathcal{C}}(id_{\mathbb{R}^3}))^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{pmatrix}$.

11. Since

$$\det \begin{pmatrix} 0 & -2 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix} = -6 \neq 0, \quad \det \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & -1 \\ 0 & 2 & 2 \end{pmatrix} = -6 \neq 0,$$

both the polynomials p_is and the q_js are 4 linearly independent vectors in $\mathbb{R}[x]_3$, hence they are two bases;

$M^{\mathcal{B}}_{\mathcal{D}}(id_{\mathbb{R}[x]_3}) = \begin{pmatrix} 3 & -4 & -5 \\ -3 & 3 & 3 \\ -1 & 3/2 & 1 \end{pmatrix}$.

12. $\text{Ker}(\varphi) = \mathcal{L}(v_2 + v_3 - v_4)$ has dim $= 1$; $\text{Im}(\varphi) = W$, that is, φ is surjective.

Please note. Remember that in general there might be more than one technique to solve the same exercise. If you find a typo, or something that you do not understand, let me know!