Linear algebra and geometry a.y. 2023-2024

Worksheet 6: exercises on chapters 12–15 from the lecture notes

(Some of these exercises come from the books by [Schlesinger], [Baldovino-Lanza], [Sernesi], [Leon])

1. Let $V = \{(a,b) \in \mathbb{R}^2 \mid a > 0, b > 0\}$ the set of ordered pairs of strictly positive real numbers, and define the following operations of addition and scalar multiplication:

   addition : $(a,b) \boxplus (c,d) := (ac, bd)$ scalar multiplication : $\alpha \boxdot (a,b) := (a^\alpha, b^\alpha)$

   Verify that:

   (a) V is not a vector space with the usual addition and scalar multiplication in $\mathbb{R}^2$.

   (b) V is a vector space over $\mathbb{R}$ if endowed with the operations (addition and scalar multiplication) defined above; find the zero vector and the opposit of the pair (a,b).

2. Define the following operations of addition and scalar multiplication on $\mathbb{R}^2$:

   addition : $(x,y) \oplus (x', y') := (x+x', y+y')$ scalar multiplication : $\alpha \otimes (x,y) := (x|\alpha|, y|\alpha|)$

   Prove that $\mathbb{R}^2$ endowed with these two operations is **not** a vector space.

3. Determine which of the following subsets of $\mathbb{R}^{3,3}$ are vector subspaces:

   (a) diagonal matrices;

   (b) upper triangular matrices;

   (c) triangular matrices (upper or lower);

   (d) matrices with $a_{11} = 0$;

   (e) matrices with$a_{12} = 1$;

   (f) skew-symmetric matrices;

   (g) matrices with $\det = 0$.

4. Let A be a fixed matrix in $\mathbb{R}^{n,n}$, and let W be the set of all matrices that commute with A:

   $$W = \{M \in \mathbb{R}^{n,n} \mid AM = MA\}.$$

   Show that W is a vector subspace of $\mathbb{R}^{n,n}$.

5. Decide whether the following sets of vectors are linearly independent or not, and whether they are generators of $\mathbb{R}^2$ or not:

   (a) $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$;

   (b) $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \end{pmatrix} \right\}$;

   (c) $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$;

   (d) $\left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix} \right\}$;

   (e) $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

6. Consider the following vectors:

$$x_1 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \qquad x = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix} \qquad y = \begin{pmatrix} -9 \\ -2 \\ 5 \end{pmatrix}$$

   (a) Is it true that $x \in \mathcal{L}(x_1, x_2)$?

   (b) Is it true that $y \in \mathcal{L}(x_1, x_2)$?

7. Let $\vec{v}_1 = 3\vec{\jmath} - 2\vec{k}$ and $\vec{v}_2 = -\vec{\imath} + 2\vec{\jmath} + 5\vec{k}$ be two vectors in S_3. For which value(s) of the parameter $\alpha \in \mathbb{R}$ does the vector $\vec{v} = -2\vec{\imath} + \vec{\jmath} + \alpha\vec{k}$ belong to the subspace $\mathcal{L}(\vec{v}_1, \vec{v}_2)$?

8. The vectors $x_1 = \vec{\imath} + 2\vec{\jmath} + 2\vec{k}$, $x_2 = 2\vec{\imath} + 5\vec{\jmath} + 4\vec{k}$, $x_3 = \vec{\imath} + 3\vec{\jmath} + 2\vec{k}$, $x_4 = 2\vec{\imath} + 7\vec{\jmath} + 4\vec{k}$, $x_5 = \vec{\imath} + \vec{\jmath}$ generate $\mathbb{R}^3$. Use the **discarding algorithm** to extract 3 linearly independent vectors from the set $\{x_1, x_2, x_3, x_4, x_5\}$ (these three linearly independent generators will thus form a basis of $\mathbb{R}^3$).

9. Which of the following sets of vectors generate the vector space $\mathbb{R}[x]_2$? Which ones are formed by linearly independent vectors? ($\mathbb{R}[x]_n$ = vector space of polynomial in the unknown x with real coefficients and degree $\leq n$)

   (a) $\{1, x^2, x^2 - 2\}$;

   (b) $\{2, x^2, x, 2x + 3\}$;

   (c) $\{x + 2, x + 1, x^2 - 1\}$;

   (d) $\{x + 2, x^2 - 1\}$.

10. Find a basis and compute the dimension of the following vector subspaces:

    (a) $W = \{(a,\ a+b,\ b-a,\ b) \mid a, b \in \mathbb{R}\} \subseteq \mathbb{R}^4$;

    (b) $U = \{(a+c,\ b-a,\ b+c) \mid a, b, c \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

11. Verify that the following subsets of $\mathbb{R}^4$ are vector subspaces, and compute their sum and intersection:

$$V_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + 2x_2 + x_3 = x_3 - x_4 = 0\}$$

$$\text{and } V_2 = \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4 \mid y_1 + y_2 + y_4 = y_3 + y_4 = 0\}.$$

12. Consider the matrix

$$A = \begin{pmatrix} -1 & 0 & 2 & 1 & 0 \\ -6 & 5 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3,5},$$

and compute the dimension of its row space, of its column space, and find bases for both.

13. Find a basis of the subspace $S = \mathcal{L}(w_1, w_2, w_3, w_4) \subseteq \mathbb{R}^4$, where:

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}, \qquad w_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \qquad w_4 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 5 \end{pmatrix}.$$

14. Given the vectors

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 3 \\ k \end{pmatrix}, \quad v_4 = \begin{pmatrix} -1 \\ k \\ 5 \end{pmatrix}, \quad v_5 = \begin{pmatrix} -1 \\ 10 \\ 13 \end{pmatrix},$$

in $\mathbb{R}^3$, find a basis of the vector subspace $\mathcal{L}(v_1, v_2, v_3, v_4, v_5)$ as the parameter $k \in \mathbb{R}$ varies.

15. Do the matrices

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

form a basis of $\mathbb{R}^{2,2}$? And what about

$$B_1 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 2 & -1 \\ -2 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 3 \\ 6 & 1 \end{pmatrix}?$$

16. Let V be a vector space over $\mathbb{R}$ of dimension 3, and let $\mathcal{B} = (v_1, v_2, v_3)$ be a basis of V. Let $U = \mathcal{L}(v_1 + v_2, v_1 - v_2)$ and $W = \mathcal{L}(v_2 + v_3, v_2 - v_3)$. Prove that $V = U + W$.

17. Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be a square matrix. The *trace* of A is the sum of the diagonal elements. In particolar, when $n = 3$:

$$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33}.$$

Prove that the subset in $\mathbb{R}^{3,3}$ of matrices with zero trace is a vector subspace, and compute its dimension.

18. Let $\mathcal{C} = (e_1, e_2, e_3)$ be the canonical basis in $\mathbb{R}^3$, and consider the following vector subspaces:

    (a) $W_1 = \mathcal{L}(e_1 + 2e_3, e_3, e_1 + e_3)$
    (b) $W_2 = \mathcal{L}(e_1, e_1 - e_2, e_1 + e_3)$
    (c) $W_3 = \mathcal{L}(e_1, 2e_2, e_1 - e_3, e_1 + 2e_2 - e_3)$

Compute a basis for each of them, and find their dimension.

19. Is it possible to find a pair of 2-dimensional subspaces U and V of $\mathbb{R}^3$ such that $U \cap V = \{0_{\mathbb{R}^3}\}$?

20. Consider the following subspaces of $\mathbb{R}[x]_2$:

$$S = \{p(x) \in \mathbb{R}[x]_2 \mid p(0) = 0\} \quad \text{and} \quad T = \{q(x) \in \mathbb{R}[x]_2 \mid q(1) = 0\}.$$

Find a basis and the dimension for:

    (a) S;
    (b) T;
    (c) $S \cap T$.

Solutions.

1. (a) V is not a vector space for several reasons: there is no zero vector, there is no opposite...
   (b) All 8 axioms of vector space are satisfied; $0_V = (1, 1)$ and $-(a, b) = (1/a, 1/b)$.

2. The reason is that $(\alpha + \beta)x \neq \alpha x + \beta x$.

3. (a) Yes;
   (b) yes;
   (c) no, it is not closed with respect to the addition;
   (d) yes;
   (e) no, there is no 0 element, and it is neither closed with respect to the addition nor the scalar multiplication;
   (f) yes;
   (g) no, it is not closed with respect to the addition.

4. Clearly, the zero matrix $0_{n,n} \in W$, because $A0_{n,n} = 0_{n,n}A = 0_{n,n}$; if $M_1, M_2 \in W$, then:
$$A(M_1 + M_2) = AM_1 + AM_2 = M_1A + M_2A = (M_1 + M_2)A,$$
so $M_1 + M_2 \in W$ and W is closed with respect to the addition. Finally, if $\lambda \in \mathbb{R}$ and $M \in W$, then:
$$A(\lambda M) = \lambda(AM) = \lambda(MA) = (\lambda M)A,$$
so $\lambda M \in W$ and W is closed with respect to scalar multiplication.

5. (a) Independent & generate;
   (b) dependent & do not generate;
   (c) dependent & generate;
   (d) dependent & do not generate;
   (e) independent & generate.

6. (a) No;
   (b) yes.

7. $\alpha = 12$

8. Starting from the first vector, one finds $\{x_1, x_2, x_5\}$, but any subset of 3 linearly independent vectors in the set $\{x_1, x_2, x_3, x_4, x_5\}$ is ok.

9. (a) Dependent & do not generate;
   (b) dependent & generate;
   (c) independent & generate;
   (d) independent & do not generate.

10. (a) A basis of W is $((1,1,-1,0),(0,1,1,1))$, hence $\dim(W)=2$;
    (b) a basis of U is $((1,-1,0),(0,1,1))$, hence $\dim(U)=2$.

11. $V_1 \cap V_2 = \{0_{\mathbb{R}^4}\}$, and $V_1+V_2 = \{(-2x_2-x_3-y_2+y_3, x_2+y_2, x_3+y_3, x_3-y_3) \mid x_2, x_3, y_2, y_3 \in \mathbb{R}\}$.

12. The row space is a subspace of dimension 3 in $\mathbb{R}^5$, with basis
$$((-1,0,2,1,0),(-6,5,2,0,1),(0,0,1,0,0));$$
the column space is a subspace of dimension 3 in $\mathbb{R}^3$, with basis
$$((1,6,0),(0,1,0),(2,2,1)).$$
Otherwise we can also remark that a dimension 3 subspace in $\mathbb{R}^3$ coincides with the whole $\mathbb{R}^3$, and take the canonical basis.

13. A basis of S is $\mathcal{B}=(w_1,w_2)$.

14. If $k \neq 4$ the first 3 vectors are linearly independent, so they form a basis and $\mathcal{L}(v_1,v_2,v_3,v_4,v_5) = \mathcal{L}(v_1,v_2,v_3)=\mathbb{R}^3$. If $k=4$ only the first 2 vectors are linearly independent.

15. (A_1,A_2,A_3,S_4) is a basis of $\mathbb{R}^{2,2}$: they are linearly independent and they generate. On the other hand, the 4 matrices B_1,B_2,B_3,B_4 are not linearly independent, and they generate a subspace of dimension 3, namely
$$\mathcal{L}(B_1,B_2,B_3,B_4)=\mathcal{L}(B_1,B_2,B_3)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c=2b\right\}.$$

16. The inclusion $\supseteq$ is trivial, because $U+W$ is a vector subspace of V. For the inclusion $\subseteq$, let $v \in V$. We know that there exist $\alpha_1,\alpha_2,\alpha_3 \in \mathbb{R}$ such that $v=\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3$; if we want to write v as an element of $U+W$, we need to find 4 coefficients a,b,c,d such that:
$$\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3 = v = a\ (v_1+v_2)+b\ (v_1-v_2)+c\ (v_2+v_3)+d\ (v_2-v_3).$$
We get the following linear system of 3 equations in the 4 unknowns a,b,c,d:
$$\begin{cases} a+b=\alpha_1 \\ a-b+c+d=\alpha_2 \\ c-d=\alpha_3 \end{cases}$$
The system has ∞^1 solutions, therefore we can always write v as the sum of an element of U and an element of W, so $V \subseteq U+W$.

17. Let $T = \{A \in \mathbb{R}^{3,3} \mid \mathrm{tr}(A) = 0\}$. One can verify that the following properties hold:

$$(i)\ \mathrm{tr}(0_{3,3}) = 0, \qquad (ii)\ \mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B), \qquad (iii)\ \mathrm{tr}(\lambda A) = \lambda \mathrm{tr}(A),$$

therefore T is a vector subspace. To compute its dimension, remark that for all $A \in T$:

$$a_{33} = -(a_{11} + a_{22}),$$

so, if $(E_{ij})_{i,j=1,\dots,3}$ is the standard basis of $\mathbb{R}^{3,3}$, we can write

$$\begin{aligned} A &= a_{11}E_{11} + a_{22}E_{22} + a_{33}E_{33} + \textstyle\sum_{\substack{i,j=1 \\ i\neq j}}^{3} a_{ij}E_{ij} \\ &= a_{11}(E_{11} - E_{33}) + a_{22}(E_{22} - E_{33}) + \textstyle\sum_{\substack{i,j=1 \\ i\neq j}}^{3} a_{ij}E_{ij}. \end{aligned}$$

Therefore the matrices of the set $\mathcal{B} = (E_{11} - E_{33}, E_{22} - E_{33}) \cup (E_{ij})_{\substack{i,j=1,\dots,3 \\ i\neq j}}$ generate T, and one can prove (try!) that they are linearly independent. All in all, $\dim_{\mathbb{R}}(T) = 9 - 1 = 8$.

You can also try to generalize this result to any size n: the subset of $\mathbb{R}^{n,n}$ of matrices with zero trace is a vector subspace of dimension $n^2 - 1$.

18. (a) Basis of $W_1 = ((1,0,2),(0,0,1))$, so $\dim W_1 = 2$
    (b) Basis of $W_2 = ((1,0,0),(1,-1,0),(1,0,1))$, so $\dim W_2 = 3$
    (c) Basis of $W_3 = ((1,0,0),(0,2,0),(0,0,-1))$, so $\dim W_3 = 3$

19. The answer is NO, and it can be proven in different ways: geometrically, 2 non-parallel planes through the origin in S_3 intersect in at least a line, not in a point. Algebraically, one can use the Grassmann formula:

$$3 = \dim(\mathbb{R}^3) \geq \dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V) = 4 - \dim(U \cap V),$$

so $\dim(U \cap V) \geq 1$.

20. (a) Basis for S: (x, x^2), $\dim S = 2$;
    (b) basis for T: $(1-x, 1-x^2)$, $\dim T = 2$;
    (c) basis for $S \cap T$: $(x^2 - x)$, $\dim(S \cap T) = 1$.

Please note. Remember that in general there might be more than one technique to solve the same exercise. If you find a typo, or something that you do not understand, let me know!