Linear algebra and geometry a.y. 2023-2024
Worksheet 5: exercises on chapters 9–11 from the lecture notes

(Some of these exercises come from the books by [Schlesinger], [Baldovino-Lanza], [Sernesi].)

1. Find cartesian equations for all lines and planes described in the exercises from worksheet 4.

2. Given the line $r : \{2x + y + z - 1 = y + 2z = 0\}$, find equations for the plane through r and:
   (a) passing through the point $A = (2, 1, 0)$;
   (b) parallel to the line $s : \{x - y - z - 2 = x + y + 2z - 1 = 0\}$;
   (c) orthogonal to the plane $\sigma : \{x + 3y - 2z = 0\}$.

3. Find equations for the plane passing through the point $P = (0, 1, 2)$ and orthogonal to the line $r : \{x + y + z = x - 2y + 3z = 0\}$.

4. In each of the following cases, find the relative position of the line r and the plane π. When they meet, find the intersection.

   (a) $r : \begin{cases} x = 1 + t \\ y = 2 - 2t \\ z = 1 - 4t \end{cases}$ $\qquad \pi : 2x - y + z = 1$

   (b) $r : \begin{cases} x = 2 - t \\ y = 1 + 2t \\ z = -1 + 3t \end{cases}$ $\qquad \pi : 2x + 2y - z = -1$

   (c) $r : \begin{cases} x + z = -1 \\ x - z = 0 \end{cases}$ $\qquad \pi : x + z = 1$

5. Verify that the lines $r : \{x + 1 = z - 2 = 0\}$ and $s : \{2x + y - 2z + 6 = y + z - 2 = 0\}$ are coplanar, and find an equation for the plane containing them.

6. Find the relative position of the plane $\pi : \{x + 5y - 3z + 5 = 0\}$ and the line $r : \{x - y - 1 = 2y - z + 2 = 0\}$.

7. Compute the distance of the point $A = (1, 2, -3)$ from the plane $\pi : \{x - 3y + 2z = 3\}$.

8. Compute the distance of the point $B = (2, -1, 0)$ from the line $r : \{x - 2z + 1 = y - z + 1 = 0\}$.

9. Compute the distance between the plane $\pi : \{2x - 3y + 6z = 14\}$ and:

   (a) the plane $\alpha : \{4x - 6y + 12z + 21 = 0\}$;
   (b) the plane $\beta : \{x - y + 12z + 1 = 0\}$;
   (c) the line $r : \{z + 1 = 2x - 3y - 8 = 0\}$.

10. Find the value(s) of the real parameter k such that the planes $\alpha_k : \{2x + ky + 4z = 4\}$, $\beta_k : \{3x + y + kz = k\}$ and $\gamma_k : \{x - ky - 4z = k - 6\}$ meet in a line.

11. Find the equation of the plane π containing the points $P = (1, 3, 1)$, $Q = (0, 4, 1)$ and perpendicular to the plane $\pi' : \{x + y + z = 37\}$.

12. Find cartesian and parametric equations of the line r passing through the point $P = (1, 2, 3)$ and orthogonal to the plane $\pi : \{x + y + 2z = 352\}$.

    Let $s : \{x + y - 3 = x - y + z - 6 = 0\}$ be another line; compute the distance between r and s and decide whether they are skew, or they meet in a point, or they are parallel or orthogonal.

13. Compute the distance between the lines

$$r : \begin{cases} x = 3 + 2t \\ y = -2 + t \\ z = 4 + 5t \end{cases} \qquad \text{and} \qquad s : \begin{cases} x = -3 + 2u \\ y = 2 - u \\ z = 7 - 3u \end{cases}$$

    and find their relative position.

Solutions.

1. ex 1: $r : \begin{cases} 3x + z = 1 \\ y = 2 \end{cases}$

   ex 2: $r : \begin{cases} 2x + z = 4 \\ 2y + z = 6 \end{cases}$

   ex 3: $\gamma : \{x - 2y + z = 0\}$

   ex 4: $\pi : \{z = 1\}$

   ex 5: $\ell : \begin{cases} x + y = 3 \\ -x + z = 2 \end{cases}$

2. (a) $2x - 3y - 7z = 1$
   (b) $2x + 4y + 7z = 1$
   (c) $2x + 4y + 7z = 1$

3. $5x - 2y - 3z + 8 = 0$

4. (a) $r \subset \pi$
   (b) $r \cap \pi = (-6, 17, 23)$
   (c) $r \parallel \pi$

5. $\pi : \{2x - 3z + 8 = 0\}$

6. $r \subseteq \pi$

7. $d(A, \pi) = \sqrt{14}$

8. $d(B, r) = \sqrt{3}$

9. (a) $d(\pi, \alpha) = 7/2$
   (b) $d(\pi, \beta) = 0$
   (c) $d(\pi, r) = 12/7$

10. $k = 2$

11. $\pi : \begin{cases} x = 1 + s - t \\ y = 3 + s + t \\ z = 1 + s \end{cases}$

12. Parametric equations: $\begin{cases} x = 1 + t \\ y = 2 + t \\ z = 3 + 2t, \end{cases}$ cartesian: $\begin{cases} x - y + 1 = 0 \\ 2x - z + 1 = 0 \end{cases}$

    r and s are skew and $d(r, s) = 4/\sqrt{5}$.

13. r and s are skew and $d(r, s) = 20/\sqrt{69}$

Please note. Remember that in general there might be more than one technique to solve the same exercise. If you find a typo, or something that you do not understand, let me know!