Linear algebra and geometry a.y. 2023-2024
Worksheet 3: exercises on chapters 6–8 from the lecture notes

1. Consider the vectors $\vec{v} = 8\vec{\imath} + 6\vec{\jmath}$ and $\vec{w} = 4\vec{\imath} - \vec{\jmath}$ in $V_2(O)$: compute their length $|\vec{v}|$ and $|\vec{w}|$, the sum $\vec{v} + \vec{w}$, the normalized vector of the sum.

2. Repeat exercise 4 using the vectors $\vec{v} = \vec{\imath} - 3\vec{\jmath} - 1\vec{k}$ and $\vec{w} = 3\vec{\imath} + \vec{k}$ in $V_3(O)$.

3. Find the value of the parameter $\alpha \in \mathbb{R}$ such that $\vec{u}$ and $\vec{v}$ are parallel, where
$$\vec{u} = \alpha\vec{\imath} - 2\vec{\jmath} - \alpha\vec{k} \quad \text{and} \quad \vec{v} = 3\vec{\imath} - 2\alpha\vec{\jmath} - 3\vec{k}.$$

4. Decide which of the following triples of vectors are coplanar:
   (a) $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 9 \\ -4 \end{pmatrix};$
   (b) $\begin{pmatrix} 5 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix};$
   (c) $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$

Exercises 5–6–7–8 are from [Baldovino-Lanza, *Algebra lineare e geometria*]:

5. Consider the vectors $\vec{v} = 2\vec{\imath} - \vec{\jmath} + 3\vec{k}$ and $\vec{u} = \vec{\imath} - \vec{\jmath} + \vec{k}$, and compute:
   (a) $2\vec{v} + 3\vec{u}$;
   (b) the angle between $\vec{u}$ and $\vec{v}$;
   (c) the vector orthogonal projection of $\vec{u}$ along the direction $\vec{v}$, and its length;
   (d) all vectors parallel to $\vec{v}$ whose orthogonal projection along the direction $\vec{u}$ has length 2.

6. Consider the vectors $\vec{u} = \vec{\jmath} - \vec{k}$ and $\vec{v} = \vec{\imath} - \vec{\jmath} + 2\vec{k}$ and compute the angle between them. Find the vector $\vec{x}$ orthogonal to $\vec{u}$ and such that $\vec{u} \times \vec{x} = \vec{u} \times \vec{v}$.

7. Which condition should two vectors $\vec{u}$ and $\vec{v}$ satisfy for the sum $\vec{u} + \vec{v}$ to be orthogonal to the difference $\vec{u} - \vec{v}$?

8. Given the vectors $\vec{v} = \vec{\imath} - \vec{\jmath} - \vec{k}$ and $\vec{u} = \vec{\imath} - 2\vec{\jmath} + \vec{k}$, decompose $\vec{v}$ into the sum of a vector $\vec{v}_{\parallel}$ parallel to $\vec{u}$ and a vector $\vec{v}_{\perp}$ orthogonal to $\vec{u}$.

Exercises 9–10 are from [Schlesinger, *Algebra lineare e geometria*]:

9. Consider the vectors $\vec{v} = 3\vec{\imath} + 5\vec{\jmath} - 3\vec{k}$ and $\vec{w}_a = 4\vec{\imath} - 2\vec{\jmath} + a\vec{k}$ in $V_3(O)$, find the values of the parameter $a \in \mathbb{R}$ such that $\vec{v} \perp \vec{w}_a$.

10. Given the points $A = (3, 2, 1)$, $B = (2, 2, 5)$, $C = (1, 2, 7)$ in S_3, write down the orthogonal projection of $\vec{AB}$ along the direction $\vec{AC}$; then compute the area of the triangle $\overset{\triangle}{ABC}$.

11. In S_3, consider the points $P = (1, 1, 1)$, $Q = (2, 1, 0)$, $R = (1-h, 1, h)$, $S = (-h+2, -h+2, 0)$, where $h \in \mathbb{R}$ is a real parameter. Find the values of h such that the volume of the tetrahedron with vertices P, Q, R, S equals $\frac{1}{3}$.

Solutions.

1. $|\vec{v}| = 10$, $|\vec{w}| = \sqrt{17}$, $\vec{v} + \vec{w} = 12\vec{\imath} + 5\vec{\jmath}$, the normalization of $\vec{v} + \vec{w}$ is $\frac{12}{13}\vec{\imath} + \frac{5}{13}\vec{\jmath}$.

2. $|\vec{v}| = \sqrt{11}$, $|\vec{w}| = \sqrt{10}$, $\vec{v} + \vec{w} = 4\vec{\imath} - 3\vec{\jmath}$, the normalization of $\vec{v} + \vec{w}$ is $\frac{4}{5}\vec{\imath} - \frac{3}{5}\vec{\jmath}$.

3. $\alpha = \pm\sqrt{3}$

4. (a) yes;
   (b) no;
   (c) no.

5. (a) $2\vec{v} + 3\vec{u} = 7\vec{\imath} - 5\vec{\jmath} + 9\vec{k}$
   (b) α such that $\cos(\alpha) = \frac{\sqrt{42}}{7}$
   (c) $\vec{u_{\parallel}} = \frac{1}{7}(6\vec{\imath} + -3\vec{\jmath} + 9\vec{k})$; $|\vec{u_{\parallel}}| = \frac{3}{7}\sqrt{14}$
   (d) $\vec{w_1} = \frac{\sqrt{3}}{3}\vec{v}$ e $\vec{w_2} = -\frac{\sqrt{3}}{3}\vec{v}$

6. $\frac{\pi}{3} + \frac{\pi}{2}$; $\vec{x} = \vec{\imath} + \frac{1}{2}\vec{\jmath} + \frac{1}{2}\vec{k}$

7. $|\vec{u}| = |\vec{v}|$

8. $\vec{v}_{\parallel} = \frac{1}{3}\vec{\imath} - \frac{2}{3}\vec{\jmath} + \frac{1}{3}\vec{k}$, $\vec{v}_{\perp} = \frac{2}{3}\vec{\imath} - \frac{1}{3}\vec{\jmath} - \frac{4}{3}\vec{k}$

9. $a = \frac{2}{3}$

10. $\vec{AB}_{\parallel} = \frac{13}{10}(-\vec{\imath} + 3\vec{k})$, area$(A\overset{\triangle}{B}C)$=1

11. $h = -1$ and $h = 3$.

Please note. Remember that in general there might be more than one technique to solve the same exercise. If you find a typo, or something that you do not understand, let me know!