Linear algebra and geometry a.y. 2023-2024

Worksheet 2: exercises on chapters 4–6 from the lecture notes

1. Reduce $A \in \mathbb{R}^{3,4}$ in row echelon form, and find its rank:

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{pmatrix}.$$

2. Reduce $B \in \mathbb{R}^{3,3}$ in row echelon form, and find its rank, as the parameters a, b, c vary in $\mathbb{R}$:

$$B = \begin{pmatrix} a & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. Decide for which values of $\alpha \in \mathbb{R}$ the matrix $C \in \mathbb{R}^{3,4}$ has rank 0, 1, 2 or 3:

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & \alpha & 2 & \alpha \\ 1 & 1+\alpha & 1 & 2\alpha \end{pmatrix}.$$

4. Study the rank of $D \in \mathbb{R}^{3,3}$, depending on the values of the parameter $k \in \mathbb{R}$:

$$D = \begin{pmatrix} 1 & k & 2k \\ k & 2k & k+1 \\ 2k & 4k & 2k \end{pmatrix}.$$

When they exist, find all solutions of the following linear systems:

5. $\begin{cases} -x + 2y - z = 2 \\ -2x + 2y + z = 4 \\ 3x + 2y + 2z = 5 \\ -3x + 8y + 5z = 17 \end{cases}$

6. $\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 1 \\ -x_1 - x_2 + 4x_3 - x_4 = 6 \\ -2x_1 - 4x_2 + 7x_3 - x_4 = 1 \end{cases}$

7. $\begin{cases} x + 2y = 4 \\ 2x + 5y - 3z = 0 \\ x + 4y - 6z = -15 \end{cases}$

8. Find the value of $\alpha \in \mathbb{R}$ such that the following linear system has a unique solution:

$$\begin{cases} x + 2y + z = 1 \\ -x + 4y + 3z = 2 \\ 2x - 2y + \alpha z = 3 \end{cases}$$

9. Consider the two linear equations $3x - y + 5z = 0$ and $x - 2y - 3z = 0$.

   (a) Figure out a possible third equation in x, y, z such that the three equations together form a homogeneous system with a unique solution (which is then the zero solution);

   (b) figure out a possible third equation in x, y, z such that the three equations together form a homogeneous system with infinitely many solutions.

   (c) Is it possible to find a third equation in x, y, z such that the three equations together form a homogeneous system with ∞^2 solutions?

10. Let A and B be $n \times n$ matrices, and let $C = A - B$. Show that if $AX_0 = BX_0$ for some column matrix $X_0 \in \mathbb{R}^{n,1}$, then $\mathrm{rk}(C) \leq n - 1$.

11. Use elementary row operations to find the inverse matrix A^{-1}, where

$$A = \begin{pmatrix} -1 & 2 \\ 3 & 3 \end{pmatrix}.$$

12. Use elementary row operations to find the inverse matrix B^{-1}, where

$$B = \begin{pmatrix} 1 & -4 & 1 \\ 2 & -6 & 5 \\ 1 & -2 & 5 \end{pmatrix}.$$

13. Compute the determinant of the matrix

$$\begin{pmatrix} 3 & 1 & -3 \\ 1 & 3 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$

in 6 different ways, performing Laplace expansion with respect to all rows and all columns, and checking that you always get the same result.

14. Compute the determinants of the following matrices, and decide whether they are invertible or not. If a matrix is invertible, compute the inverse matrix.

   (a) $A = \begin{pmatrix} 3 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$

(b) $B = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 5 \\ 2 & 1 & 2 \end{pmatrix}$

(c) $C = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{pmatrix}$

(d) $D = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{pmatrix}$

15. Find all values of $\alpha \in \mathbb{R}$ such that the determinant of the following matrix is zero:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 9 & \alpha \\ 1 & \alpha & 3 \end{pmatrix}$$

16. Let $A, B \in \mathbb{R}^{3,3}$ such that $\det(A) = 4$ and $\det(B) = 5$. Compute:

(a) $\det(AB)$

(b) $\det(3A)$

(c) $\det(2AB)$

(d) $\det(A^{-1}B)$

17. Let $A \in \mathbb{R}^{n,n}$ be a skew-symmetric matrix, with $n = 2k + 1$ an odd number; explain why $\det(A) = 0$.

Solutions.

1. $\text{rk}(A) = 2$, and a row reduced matrix that one can obtain from A with elementary row operations is A':
$$A' = \begin{pmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Remark that B is already in row echelon form. If $a \neq 0$, we can count 3 pivots, and $\text{rk}(B) = 3$. If instead $a = 0$, independently from the fact that b and c are zero or not, the matrix has 2 pivots, so$\text{rk}(B) = 2$.

3. If $\alpha = 0, 1$, $\text{rk}(C) = 2$, otherwise $\text{rk}(C) = 3$.

4. If $k = 0, 2$, $\text{rk}(D) = 2$, otherwise $\text{rk}(D) = 3$.

5. The system has unique solution $(0, 3/2, 1)$.

6. The system has ∞^1 solutions, that can be written, for example, in the form:
$$(2 - 6t, 4 + t, 3 - t, t),\ t \in \mathbb{R}.$$

7. The system is not compatible.

8. $\alpha \neq -2$.

9. (a) Any equation such that the 3×3 matrix of coefficients has rank 3 will work: for example, $3x - y = 0$;
   (b) any equation such that the 3×3 matrix of coefficients has rank ≤ 2 will work: for example, $3x - y + 5z = 0$.
   (c) No, it's not possible, because the rank of any matrix of coefficients will be ≥ 2.

10. Using the properties of matrix addition and multiplication:
$$AX_0 = BX_0 \quad \Leftrightarrow \quad AX_0 - BX_0 = 0_{n,1} \quad \Leftrightarrow \quad (A - B)X_0 = 0_{n,1} \quad \Leftrightarrow \quad CX_0 = 0_{n,1},$$
so the homogeneous linear system $CX = 0$ has a nonzero solution: necessarily, $\text{rk}(C) \leq n - 1$.

11. $A^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & \frac{1}{9} \end{pmatrix}$.

12. $B^{-1} = \begin{pmatrix} -10 & 9 & -7 \\ -\frac{5}{2} & 2 & -\frac{3}{2} \\ 1 & -1 & 1 \end{pmatrix}$

13. The determinant is -5; below is for example the expansion with respect to the first row:

$$\det \begin{pmatrix} 3 & 1 & -3 \\ 1 & 3 & 0 \\ 1 & 2 & -1 \end{pmatrix} = 3 \cdot \det \begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} + (-3) \cdot \det \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} = -9+1+3 = -5.$$

14. (a) $\det(A) = -3$, A invertible, $A^{-1} = \begin{pmatrix} 1/3 & 7/3 & -5/3 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{pmatrix}$;
    (b) $\det(B) = 2$, B invertible, $B^{-1} = \begin{pmatrix} 1/2 & -1/2 & 1 \\ 1 & 1 & -3 \\ -1 & 0 & 1 \end{pmatrix}$;
    (c) $\det(C) = 0$, not invertible;
    (d) $\det(D) = 0$, not invertible.

15. $\alpha = 5, -3$.

16. (a) 20,
    (b) 108,
    (c) 160,
    (d) 5/4.

17. The reason is that $\det(A) = \det({}^tA)$, and $\det(\lambda A) = \lambda^n \det(A)$: if A is skew-symmetric, then $\det(A) = \det({}^tA) = \det(-A) = (-1)^n \det(A)$. If n is odd, $(-1)^n = -1$, so the chain of equalities reads $\det(A) = -\det(A)$, that in turn implies $\det(A) = 0$.

Please note. Remember that in general there might be more than one technique to solve the same exercise. If you find a typo, or something that you do not understand, let me know!