Linear algebra and geometry a.y. 2023-2024

Worksheet 10: exercises on chapters 22-23 from the lecture notes

(Some of these exercises come from the books by [Schlesinger], [Baldovino-Lanza], [Sernesi], [Leon])

1. Consider the following pencil of conics:

$$\mathcal{C}_t: \quad x^2 + (1-t)y^2 + 2tx - 2(1-t)y + 2 - t = 0,$$

and find the values of the parameter t such that $\mathcal{C}_t$ is a

(a) parabola;
(b) hyperbola;
(c) ellipse;
(d) circle;
(e) degenerate conic.

2. Classify the following conics:

(a) $2x^2 + 2xy + x + 5y - 10 = 0$;
(b) $3x^2 - 8xy - 3y^2 + 10 = 0$;
(c) $9x^2 + 16y^2 + 24xy - 40x + 30y = 0$;
(d) $3x^2 + 2xy + 3y^2 + 2\sqrt{2}x - 2\sqrt{2}y = 0$.

3. Find all degenerate conics in the family

$$\mathcal{F}: 2\alpha x^2 + 2y^2 + 4\alpha x + 2y + 2\alpha = 0,$$

where $\alpha \in \mathbb{R}$ is a real parameter.

4. Describe the conics in the family

$$x^2 - 4xy + y^2 + 7h^2 + 1 = 0$$

as the parameter $h \in \mathbb{R}$ varies.

5. Verify that the equation $xy - 2x + y - 3 = 0$ represents an equilateral hyperbola in the plane, finding its canonical form and the rototranslation one should apply to get it.

Solutions.

1. (a) Never;
   (b) $t > 1$;
   (c) $t < 1$, $t \neq -1$;
   (d) $t = 0$ (imaginary circle);
   (e) $t = \pm 1$.

2. (a) Two non parallel lines meeting in a point;
   (b) (equilateral) hyperbola;
   (c) parabola;
   (d) ellipse.

3. $y^2 + y = 0$ and $x^2 - 3y^2 + 2x - 3y + 1 = 0$.

4. The conic is a hyperbola for all values of $h \in \mathbb{R}$.

5. The canonical equation of the conic is $\frac{x'^2}{2} - \frac{y'^2}{2} = 2$ which is precisely the canonical form of the equilateral hyperbola. This can be found by applying the rototranslation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} 1/\sqrt{2} \\ 3/\sqrt{2} \end{pmatrix}.$$

Please note. Remember that in general there might be more than one technique to solve the same exercise. If you find a typo, or something that you do not understand, let me know!