Linear algebra and geometry a.y. 2023-2024

Worksheet 1: exercises on chapters 1–3 from the lecture notes

1. Consider the matrices
$$A = \begin{pmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{pmatrix} \in \mathbb{R}^{3,2} \qquad \text{and} \qquad B = \begin{pmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{pmatrix} \in \mathbb{R}^{2,3},$$
and compute the products AB and BA. Is it possible to compute A^2 and B^2?
Then compute the transposes ${}^tA$, ${}^tB$, ${}^t(AB)$ and ${}^t(BA)$.

2. Compute $A^2 - {}^tA + I_3$, where
$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & \frac{1}{2} \\ 0 & -2 & -1 \end{pmatrix} \in \mathbb{R}^{3,3}.$$

3. For which value of $a \in \mathbb{R}$ is the matrix $M = \begin{pmatrix} 0 & 2 \\ a & 1 \end{pmatrix}$ symmetric? And skew-symmetric?

4. Let $A \in \mathbb{R}^{n,n}$ be a symmetric matrix, and suppose that A is invertible; is it true that the inverse A^{-1} is also symmetric? And what about the same statement for skew-symmetric matrices?

5. Consider the matrices
$$A = \begin{pmatrix} \alpha & \alpha - 1 \\ \beta & \beta - 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \alpha - 1 & \alpha - 1 \\ \beta & \beta - 2 \end{pmatrix}$$
in $\mathbb{R}^{2,2}$: find the values of the parameters α and β such that the relation $AB = 0_{2,2}$ holds.

6. Let
$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \in \mathbb{R}^{2,2};$$
compute A^2 and A^3. Can you guess what will A^n turn out to be for n any positive integer?

7. Write down the matrix of coefficients, the matrix of constant terms, and the augmented matrix associated to the following linear systems:

   (a) $\begin{cases} x_1 - x_2 + 3x_3 + x_4 = 5 \\ 6x_1 - x_3 = 2 \end{cases}$

(b) $\begin{cases} 3x_1 + 2x_2 - x_3 = -2 \\ -3x_1 - x_2 + x_3 = 5 \\ 3x_1 + 2x_2 + x_3 = 2 \end{cases}$

(c) $\begin{cases} x + \sqrt{3}y = z + 2 \\ -y + x = 0 \\ x + y - z + 4 = 0 \end{cases}$

8. Write down the linear systems whose associated augmented matrices are:

(d) $(A_d|B_d) = \left(\begin{array}{cccc|c} 1 & \sqrt{3} & -1 & -2 & 3 \\ 0 & 0 & 1 & 1/2 & 7 \\ 2 & 0 & 0 & 2 & 0 \end{array}\right)$

(e) $(A_e|B_e) = \left(\begin{array}{cc|c} 5 & -1 & 0 \\ 5 & 1 & 3 \\ 0 & 1/3 & 1/2 \end{array}\right)$

(f) $(A_f|B_f) = \left(\begin{array}{ccccc|c} 0 & 2 & -1 & 1 & 7 & 1/5 \\ 1 & 2 & 1 & 1/2 & 0 & -5 \end{array}\right)$

Solutions.

1.

$$AB = \begin{pmatrix} -14 & 1 & -3 \\ 12 & 6 & 30 \\ -14 & -2 & -15 \end{pmatrix} \in \mathbb{R}^{3,3}, \qquad BA = \begin{pmatrix} -1 & -1 \\ 20 & -22 \end{pmatrix} \in \mathbb{R}^{2,2};$$

the matrices A^2 and B^2 do not exist.

$${}^tA = \begin{pmatrix} 3 & 2 & 1 \\ -2 & 4 & -3 \end{pmatrix} \in \mathbb{R}^{2,3} \qquad {}^tB = \begin{pmatrix} -2 & 4 \\ 1 & 1 \\ 3 & 6 \end{pmatrix} \in \mathbb{R}^{3,2},$$

$${}^t(AB) = \begin{pmatrix} -14 & 12 & -14 \\ 1 & 6 & -2 \\ -3 & 30 & -15 \end{pmatrix} = {}^tB{}^tA \in \mathbb{R}^{3,3}, \qquad {}^t(BA) = \begin{pmatrix} -1 & 20 \\ -1 & -22 \end{pmatrix} = {}^tA{}^tB \in \mathbb{R}^{2,2}.$$

2. $A^2 - {}^tA + I_3 = \begin{pmatrix} 1 & 5 & 1/2 \\ -1 & 2 & 5/2 \\ 1 & -5/2 & 2 \end{pmatrix}.$

3. The only value for which $M = {}^tM$ is $a = 2$. On the other hand, a skew-symmetric ($M = -{}^tM$) has all diagonal entries equal to 0, so there is no value of a such that M is skew-symmetric.

4. Both statements are true, because ${}^t(A^{-1}) = ({}^tA)^{-1}$; so if $A = {}^tA$ then ${}^t(A^{-1}) = ({}^tA)^{-1} = A^{-1}$, and if $A = -{}^tA$ then ${}^t(A^{-1}) = ({}^tA)^{-1} = (-A)^{-1} = -(A^{-1})$.

5. $\alpha = \beta = 1$.

6. $A = A^2 = A^3 = A^n$ for all $n \in \mathbb{Z}$, $n \geq 1$.

7. (a) $A_a = \begin{pmatrix} 1 & -1 & 3 & 1 \\ 6 & 0 & -1 & 0 \end{pmatrix}, \quad B_a = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad (A_a|B_a) = \left(\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 5 \\ 6 & 0 & -1 & 0 & 2 \end{array}\right);$

(b) $A_b = \begin{pmatrix} 3 & 2 & -1 \\ -3 & -1 & 1 \\ 3 & 2 & 1 \end{pmatrix}, \quad B_b = \begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix}, \quad (A_b|B_b) = \left(\begin{array}{ccc|c} 3 & 2 & -1 & -2 \\ -3 & -1 & 1 & 5 \\ 3 & 2 & 1 & 2 \end{array}\right);$

(c) $A_c = \begin{pmatrix} 1 & \sqrt{3} & -1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad B_c = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}, \quad (A_c|B_c) = \left(\begin{array}{ccc|c} 1 & \sqrt{3} & -1 & 2 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -4 \end{array}\right);$

8. (d) $\begin{cases} x_1 + \sqrt{3}x_2 - x_3 - 2x_4 = 3 \\ x_3 + 1/2x_4 = 7 \\ 2x_1 + 2x_4 = 0 \end{cases}$

(e) $\begin{cases} 5x_1 - x_2 = 0 \\ 5x_1 + x_2 = 3 \\ 1/3x_2 = 1/2 \end{cases}$

(f) $\begin{cases} 2_2 - x_3 + x_4 + 7x_5 = 1/5 \\ x_1 + 2x_2 + x_3 + 1/2x_4 = -5 \end{cases}$

Please note. Remember that in general there might be more than one technique to solve the same exercise. If you find a typo, or something that you do not understand, let me know!