Lecture 6

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Lecture plan

- Lecture 1: introduction, compression spaces, primitive spaces.
- Lecture 2: spaces of matrices of constant and bounded rank, sheaves and vector bundles; dimension bounds.
- Lecture 3: spaces of matrices and min. free resolution of graded modules.
- Lecture 4: the case of symmetric matrices; applications to diff. geometry.
- Lecture 5: the case of skew-symm. matrices; applications to PDEs.
- **Lecture 6: applications to numerical analysis: uniform determinantal representations and compression spaces.**

In this lecture we get to see spaces of matrices of bounded rank “in action”, that is, as a tool to solve a numerical analysis problem.

Our main reference is the paper

[A.B., J. van Doornmalen, J.Draisma, M. E.Hochstenbach, and B. Plestenjak, *Uniform Determinantal Representations*, SIAM Journal on Applied Algebra and Geometry 1 (2017)].

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Fix two positive integers $d, n \in \mathbb{Z}_{\geq 0}$, and consider an

n-variate polynomial p of degree at most d, with coefficients in a field K.

Definition

- A determinantal representation of p is an $N \times N$ matrix M of the form

$$M = A_0 + \sum_{i=1}^{n} x_i A_i,$$

  where $A_i \in K^{N \times N}$, and such that $\det(M) = p$.
- The integer N is the size of the determinantal representation.
- The determinantal complexity of p is the minimal size of any of its determinantal representations.

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Example

A smooth cubic curve in $\mathbb{P}^2$ of equation

$$p(x,y,z) = y^2z - x(x-5z)(x+7z) = 0$$

has determinantal representation

$$p(x,y,z) = \det\begin{pmatrix} -x & -y & 0 \\ 0 & x-5z & y \\ z & 0 & x+7z \end{pmatrix}.$$

Determinantal representations of polynomials play a fundamental role in several mathematical areas. Among them:

- *Optimization*, where one is particularly interested in the case where $K = \mathbb{R}$.

- *Complexity theory*, where a central role is played by Valiant's conjecture that the permanent of an $m \times m$ matrix does not admit a determinantal representation of size polynomial in m.

(For a good list of references, I refer to our paper.)

As you know, I am most familiar with:

• *algebraic geometry*, where it is known that each plane curve of degree d over an algebraically closed field K admits a determinantal representation of size d: the classical paper

[A. C. Dixon, *Note on the reduction of a ternary quartic to a symmetrical determinant*, Proc. Cambridge Philos. Soc., 11 (1900-1902)]

contains the following result:

Theorem

Every plane curve $p(x,y,z) = 0$ has a determinantal representation

$$p(x,y,z) = c \cdot \det(xA + yB + zC).$$

Even more, one can choose A, B, C symmetric.

For $n \geq 3$, only certain hypersurfaces have a determinantal representation of size equal to their degree.

Again Dixon proved the following result in the paper

[L. E. Dickson, *Determination of all general homogeneous polynomials expressible as determinants with linear elements*, Trans. Amer. Math. Soc., 22 (1921)]

Theorem

A generic homogeneous polynomial in $n+1$ variables of degree d has a determinantal representation if and only if

1. $n = 2$ *(curves);*
2. $n = 3$ *and* $d = 2, 3$ *(surfaces);*
3. $n = 4$ *and* $d = 2$ *(threefolds).*

Remark. While the general hypersurface in $\mathbb{P}^n$ does not admit a determinantal representation, unless it is one of the cases above, its higher multiples do: from the theory of generalized Clifford algebras [Backelin-Herzog-Sanders, 1986] proved that for every polynomial p there exists a k such that p^k has a determinantal representation.

Some excellent references:

[A. Beauville, *Determinantal hypersurfaces*, Michigan Math. J., 48 (2000)],

[I. Dolgachev, *Classical Algebraic Geometry: A Modern View*, Cambridge University Press (2012)],

[V. Vinnikov, *Self-adjoint determinantal representations of real plane curves*, Mathematische Annalen (1993)].

Recently determinantal representations of polynomials have been proposed in *numerical analysis and scientific computing* for efficiently solving systems of equations in the paper

[B. Plestenjak and M. E. Hochstenbach, *Roots of bivariate polynomial systems via determinantal representations*, SIAM J. of Scientific Computing, 38 (2016)],

and this has been our starting point.

⚠ For this application, it is crucial to have determinantal representations **not** of a single polynomial p, but rather of all n-variate polynomials of degree at most d.

Moreover, the representation should be easily computable from the coefficients of p: specifically, we look for determinantal representations in which the entries of the matrices A_i themselves depend affine-linearly on the coefficients of p.

Example

The determinantal representation of a cubic curve in $\mathbb{P}^2$ is a good example.

On the contrary, the representation

$$-y^2z + x^3 + \alpha xz^2 + \beta z^3 = \det \begin{pmatrix} x + \frac{1}{2}tz & y + sz & (\alpha + \frac{3}{4}t^2)z \\ 0 & x - tz & y - sz \\ -z & 0 & x + \frac{1}{2}tz \end{pmatrix}$$

with $s^2 = t^3 + \alpha t + \beta$, is not affine linear in the coefficients.

More precisely, let

- K be a field and fix two integers $d, n \in \mathbb{Z}_{\geq 0}$.
- Let F_d denote the polynomials of degree at most d in the polynomial ring $K[x_1, \ldots, x_n]$, and
- let $p_{n,d}$ be the generic polynomial of that degree, of the form

$$p_{n,d} = \sum_{|\alpha| \leq d} c_\alpha x^\alpha$$

where $x = (x_1, \ldots, x_n)$, $\alpha \in \mathbb{Z}^n_{\geq 0}$, $|\alpha| = \sum_i \alpha_i$, $x^\alpha = \prod_i x_i^{\alpha_i}$, and where we consider c_α as a variable for each α.

Definition

A uniform determinantal representation of $p_{n,d}$ is an $N \times N$ matrix M with entries from $K[(x_1 \ldots, x_n), (c_\alpha)_{|\alpha| \leq d}]$, of **degree at most 1 in each of these two sets of variables**, such that $\det(M) = p_{n,d}$.

Example (The bivariate quadric)

The identity

$$a + bx + cy + dx^2 + exy + fy^2 = \det \begin{pmatrix} -x & 1 & 0 \\ -y & 0 & 1 \\ a & b + dx + ey & c + fy \end{pmatrix}$$

shows a uniform determinantal representation of size 3 of the generic bivariate quadric. (So here $n = d = 2$, and $N = 3$.)

In applications, the matrix M is used as input to algorithms in numerical linear algebra that scale unfavorably with matrix size N.

It is therefore natural to ask the following fundamental

Question

What is the minimal size $N^*(n, d)$ of any uniform determinantal representation of the generic polynomial of degree d in n variables?

A construction by [Plestenjak-Hochstenbach] shows that for fixed $n = 2$ and $d \to \infty$ we have

$$\boxed{N^*(2, d) \leq \frac{1}{2}d^2 + O(d).}$$

We improved this construction giving two interesting uniform determinantal representations of bivariate polynomials of size $2d + 1$, and $2d - 1$, that we will now see in detail.

Remark. In view of the obvious lower bound of d this is clearly sharp up to a constant factor for $d \to \infty$, although we do not know where in the interval $[d, 2d - 1]$ the true answer lies.

Example (Construction of a uniform determinantal representations of bivariate polynomials of size $2d+1$)

Let $p = \sum_{i+j\leq 4} c_{ij}x^i y^j$ be the generic polynomial of degree $d = 4$ in 2 variables. It has the following uniform determinantal representation:

$$p = \det \begin{pmatrix} -x & 1 & & & & & & & \\ & -x & 1 & & & & & & \\ & & -x & 1 & & & & & \\ & & & -x & 1 & & & & \\ c_{00} & c_{10} & c_{20} & c_{30} & c_{40} & -y & & & \\ c_{01} & c_{11} & c_{21} & c_{31} & & 1 & -y & & \\ c_{02} & c_{12} & c_{22} & & & & 1 & -y & \\ c_{03} & c_{13} & & & & & & 1 & -y \\ c_{04} & & & & & & & & 1 \end{pmatrix}.$$

This example extends to a uniform determinantal representation of size $2d+1$ for the generic bivariate polynomial p of degree d. We get $p = \det(M)$, where

$$M = (-1)^d \begin{pmatrix} M_x & 0 \\ L & M_y^T \end{pmatrix}.$$

Here M_x and M_y are $d \times (d+1)$ matrices with 1 on the first upper diagonal and $-x$ and $-y$, respectively, on the main diagonal:

$$M_x = \begin{pmatrix} -x & 1 & & & \\ & -x & 1 & & \\ & & \ddots & \ddots & \\ & & & -x & 1 \end{pmatrix}, \qquad M_y = \begin{pmatrix} -y & 1 & & & \\ & -y & 1 & & \\ & & \ddots & \ddots & \\ & & & -y & 1 \end{pmatrix}.$$

The matrix L instead is a $(d+1) \times (d+1)$ triangular matrix such that

$$\ell_{ij} = c_{j-1,i-1} \qquad \text{for} \quad i+j \leq d+2$$

and 0 otherwise:

$$L = \begin{pmatrix} c_{00} & c_{10} & \cdots & c_{d-1,0} & c_{d0} \\ c_{01} & c_{11} & \cdots & c_{d-1,1} & \\ \vdots & \vdots & & & \\ c_{0,d-1} & c_{1,d-1} & & & \\ c_{0d} & & & & \end{pmatrix}.$$

Example (Construction of a uniform determinantal representations of bivariate polynomials of size $2d-1$)

The previous example can be slightly improved to a representation of size $2d-1$, with the same construction but taking as “building block” the following (again for $d=4$ and $n=2$):

$$\begin{pmatrix}
-x & 1 & & & & & \\
 & -x & 1 & & & & \\
 & & -x & 1 & & & \\
c_{00} & c_{10} & c_{20} & c_{30}+c_{40}x & -y & & \\
c_{01} & c_{11} & c_{21}+c_{31}x & & 1 & -y & \\
c_{02}+c_{03}y & c_{12}+c_{22}x & & & & 1 & -y \\
c_{13}x+c_{04}y & & & & & & 1
\end{pmatrix}.$$

Inspired by the bivariate polynomial construction, we also built an algorithm that constructs a determinantal representation for an n-variate polynomial of degree at most d, where $n \geq 2$.

It is based on the construction from the proof of our main Theorem (that will come later!) for even n, but can be applied to odd n as well.

For large d the algorithm returns matrices of size $\mathcal{O}(d^{\lceil n/2 \rceil})$.

In the worst case, when all coefficients in p are nonzero, the overall complexity is $\mathcal{O}\left(\binom{n+d}{n} d\right)$.

Let $m = \lfloor n/2 \rfloor$ and let S_1 be the list of all monomials in $x_1, \ldots, x_m$ of degree $d-1$ and S_2 the list of all monomials in $x_{m+1}, \ldots, x_n$ of degree $d-1$.

We take for V_i the span of all monomials in S_i for $i = 1, 2$.

The algorithm returns an $N \times N$ block matrix

$$M = \begin{bmatrix} M_{V_1} & 0 \\ L & M_{V_2}^T \end{bmatrix}$$

such that $\det(M) = \pm p$, where

- M_{V_1} is of size $(N_1 - 1) \times N_1$, $N_1 = \binom{m+d-1}{m}$,
- M_{V_2} is of size $(N_2 - 1) \times N_2$, $N_2 = \binom{n-m+d-1}{n-m}$, and
- L is of size $N_2 \times N_1$, $N = N_1 + N_2 - 1$.

Example

If we apply the algorithm to

$$\boxed{p = 2 + 3x_1^2x_2x_3 + 4x_1x_2x_3 + 5x_2^2x_4 + 6x_2x_3x_4 + 7x_3x_4 + 8x_5^4,}$$

then $n = 5$, $d = 4$, and the algorithm uses monomial lists (ordered in the degree negative lexicographic ordering)

$$S_1 = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\},$$

$$S_2 = \{1, x_3, x_4, x_5, x_3^2, x_3x_4, x_3x_5, x_4^2, x_4x_5, x_5^2, x_3^3, x_3^2x_4, x_3^2x_5, \dots, x_4^3, x_4^2x_5, x_4x_5^2, x_5^3\}.$$

The final result is a 29×29 matrix $M = \begin{bmatrix} M_{V_1} & 0 \\ L & M_{V_2}^T \end{bmatrix}$ that satisfies

$$\det(M) = -p,$$

where:

$$M_{V_1} = \begin{bmatrix}
-x_1 & 1 & & & & & & & & \\
-x_2 & & 1 & & & & & & & \\
 & -x_1 & & 1 & & & & & & \\
 & & -x_1 & & 1 & & & & & \\
 & & -x_2 & & & 1 & & & & \\
 & & & -x_1 & & & 1 & & & \\
 & & & & -x_1 & & & 1 & & \\
 & & & & & -x_1 & & & 1 & \\
 & & & & & -x_2 & & & & 1
\end{bmatrix},$$

M_{V_2} is a 19×20 matrix with the following nonzero elements:

a) 1 on the first upper diagonal,
b) $-x_3$ on $(1,1),(4,2),(5,3),(6,4),(10,5),(11,6),(12,7),(13,8),(14,9)$, and $(15,10)$,
c) $-x_4$ on $(2,1),(7,3),(8,4),(16,8),(17,9)$, and $(18,10)$,
d) $-x_5$ on $(3,1),(9,4)$, and $(19,10)$,

and finally L is a 20×10 matrix with nonzero elements

$$\ell_{11} = 2, \quad \ell_{25} = 4, \quad \ell_{36} = 5, \quad \ell_{61} = 7, \quad \ell_{63} = 6 + 3x_1, \quad \text{and} \quad \ell_{20,1} = 8x_5.$$

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So what do spaces of singular matrices have to do with all this?

Remark that we can decompose M as $M_0 + M_1$, where M_0 contains all terms in M that do not contain any c_α, and where M_1 contains all terms in M that do.

Key Lemma

For any uniform determinantal representation $M = M_0 + M_1$ of size N, the determinant of M_0 is the zero polynomial in $K[x_1, \ldots, x_n]$.

Moreover, at every point $\bar{x} \in K^n$, the rank of the specialization $M_0(\bar{x}) \in K^{N \times N}$ is exactly $N - 1$.

The first statement follows from the fact that $\det(M_0)$ is the part of the polynomial $\det(M)$ which is homogeneous of degree zero in the c_α; hence zero. The second statement is not hard to prove, but it takes a little more patience.

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Definition

A subspace $\mathcal{A} \subseteq K^{N\times N}$ is called a compression space if there exists a subspace $U \subseteq K^N$ with $\dim(\langle u^T A \mid A \in \mathcal{A}, u \in U\rangle_K) < \dim U$. We call the space U a witness for the singularity of $\mathcal{A}$.

Given any two subspaces $U, V \subseteq K^N$ with $\dim V = \dim U - 1$, the space of all matrices which map U into V (acting on row vectors) is a compression space with witness U. It is easy to see that these spaces are inclusion-wise maximal among all singular spaces.

Exercise. Prove the equivalence of Eisenbud-Harris' definition of compression space with the one above.

Example

In the example with $n = 2$ and $d = 4$ that we saw previously, one has $M = M_0 + M_1$, where

$$M_0 = \begin{pmatrix} -x & 1 & & & & & & & \\ & -x & 1 & & & & & & \\ & & -x & 1 & & & & & \\ & & & -x & 1 & & & & \\ & & & & & -y & & & \\ & & & & & 1 & -y & & \\ & & & & & & 1 & -y & \\ & & & & & & & 1 & -y \\ & & & & & & & & 1 \end{pmatrix}.$$

In this case, M_0 represents a compression space with witness $U = \langle e_5, \dots, e_9 \rangle_K$, which is mapped into $\langle e_6, \dots, e_9 \rangle_K$.

Theorem

For fixed n, there exists a determinantal representation $M = M_0 + M_1$ *of the generic n-variate polynomial of degree d of size* $\frac{1}{n \cdot n!} d^n + O(d^{n-1})$ *such that the singular matrix space represented by* M_0 *is a compression space with a one-dimensional witness.*
Moreover, under this latter additional condition on M_0*, the bound is sharp.*

As a corollary, we can give several uniform representations of, to the best of our knowledge, the smallest possible size for cases where n and d are small.

Moreover we compute the effective value of $N^*(n, d)$ for small values of n and d:

Theorem

- $N^*(2, 2) = 3$
- $N^*(3, 2) = 4$

The proofs use the classification of spaces of small singular matrices in an essential manner, as well as the action of $AGL_n(K)$ on uniform determinantal representations:

let $g \in AGL_n(K)$ be an affine transformation of K^n, and expand

$$p_{n,d}(g^{-1}x, c) = \sum_{|\alpha| \leq d} c'_\alpha x^\alpha.$$

Now let $M = M(x, c)$ be a uniform determinantal representation of $p_{n,d}$. Then

$$\det(M(g^{-1}x, \rho(g)^{-1}c)) = p_{n,d}(g^{-1}x, \rho(g)^{-1}c) = p_{n,d}(x, c),$$

i.e., $M(g^{-1}x, \rho(g)^{-1}c)$ is another uniform determinantal representation of $p_{n,d}$.

The action of g is given by $M \mapsto M(g^{-1}x, \rho(g)^{-1}c)$.

Conjecture: $N^*(4,2) = 5$.

Remark. It should be possible to prove it in a similar manner, using the classification of 4×4-singular matrix spaces from

[P. Fillmore, C. Laurie, and H. Radjavi, *On matrix spaces with zero determinant*, Linear and Multilinear Algebra, 18 (1985)].

On the other hand, recall from the first lecture that for $N \geq 5$ there are infinitely many conjugacy classes of inclusionwise maximal singular $N \times N$-matrix spaces: therefore fundamentally new ideas will be needed to prove lower bounds in larger situations.

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In the following table the minimal sizes known to us of uniform determinantal representations for some small values of n and d are listed:

n	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
2	3	5	7	9	11	13	15	17
3	4	7	10	14	18	22	27	34
4	5	9	14	19	26	34	44	
5	6	11	18	26				
6	7	13	22	33				
7	8	15	27	39				
8	9	17	32					

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To conclude, in our project, among other things, we also study the asymptotic behavior of $N^*(n,d)$ for fixed n and $d \to \infty$. The main result is the following:

Theorem

For fixed $n \in \mathbb{Z}_{\geq 2}$ there exist positive constants C_1, C_2 (depending on n) such that for each $d \in \mathbb{Z}_{\geq 0}$

$$C_1 d^{n/2} \leq N^*(n,d) \leq C_2 d^{n/2}.$$

Moreover, C_1 can be chosen such that the determinantal complexity of any sufficiently general polynomial is at least $C_1 d^{n/2}$.

Here "sufficiently general" means that the coefficient vector of the polynomial lies in some (unspecified) Zariski-open and dense subset (over infinite fields), or should be interpreted in a suitable counting sense (over finite fields).

There are still many interesting open questions.

- First of all, in a situation where the degree d is fixed and the number n of variables grows, what is the asymptotic behaviour of $N^*(n, d)$?

- Second, in the case of fixed n and varying d that we studied, what are the best constants C_1 and C_2 in our main Theorem? More specifically, for fixed n, does $\lim_{d\to\infty} \frac{N^*(n,d)}{d^{n/2}}$ exist, and if so, what is its value?

- Third, how can our techniques for upper bounds and lower bounds be further sharpened? Can singular matrix spaces *other* than compression spaces be used to obtain tighter upper bounds (constructions) on $N^*(n, d)$? Can the action of the affine group be used more systematically to find lower bounds on $N^*(n, d)$?