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Lecture 5

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August 17-29, 2019

DISMA Dipartimento di Scienze Matematiche G. L. Lagrange
ECCELLENZA 2018 · 2022

Lecture plan

- Lecture 1: introduction, compression spaces, primitive spaces.
- Lecture 2: spaces of matrices of constant and bounded rank, sheaves and vector bundles; dimension bounds.
- Lecture 3: spaces of matrices and min. free resolution of graded modules.
- Lecture 4: the case of symmetric matrices; applications to diff. geometry.
- **Lecture 5: the case of skew-symm. matrices; applications to PDEs.**
- Lecture 6: applications to numerical analysis: uniform determinantal representations and compression spaces.

The co-rank 2 case

For both symmetric and skew-symmetric spaces of matrices, the easiest case to study is that of co-rank 2, simply because of the numerous extra-properties that rank-2 vector bundles on $\mathbb{P}^2$ satisfy.

For example, we have seen that if the space A is symmetric or skew-symmetric then the kernel bundle K and the cokernel bundle N are linked by the relation $N \simeq K^\vee(1)$.

Now, if F has rank 2, then $F^\vee \simeq F(-c_1(F))$, so that the relation above reads

$$N \simeq K(-\tfrac{r}{2} + 1).$$

In this case Westwick's bounds on the dimension read:

$$3 \leq d(r, r+2) \leq 5,$$

and in fact [Ilic-Landsberg]'s result completely solves the problem for the symmetric case, where

$$d_{sym}(r, r+2) = 3,$$

the minimal possible value.

The analogous question

$$\boxed{\text{what about } d_{skew}(r, r+2)?}$$

is still open.

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The skew-symmetric case, co-rank 2

We will now take a look at the skew-symmetric co-rank 2 case, from partial results to the state of the art.

Once again, the subject has roots in classical algebraic geometry, since such spaces of skew-symmetric matrices of rank $r = 2s$ correspond to linear subspaces in the $s - th$ secant variety of the Grassmannian $\mathbb{G}(2, n)$ of 2-dimensional subspaces in an n-dimensional space (or lines in $\mathbb{P}^{n-1}$ if you prefer the projective version) that do not meet the $s - 1$-th secant:

$$\sigma_s(\mathbb{G}(2, n)) \setminus \sigma_{s-1}(\mathbb{G}(2, n)).$$

A bit of history/references:

- [L.Manivel and E.Mezzetti, *On linear spaces of skew-symmetric matrices of constant rank*, Manuscripta Mathematica 117 (2005)] contains a complete classification of spaces of skew-symmetric matrices of size 6 and constant rank 4, together with a description of the associated vector bundles. In particular, $d_{skew}(4,6) = 3$.

- [M.L.Fania and E.Mezzetti, *Vector spaces of skew-symmetric matrices of constant rank*, Linear Algebra and its Applications 434 (2011)]; contains a similar (but not complete) classification of spaces of skew-symmetric matrices of size 8 and constant rank 8, and a description of the associated vector bundle, and on the immersion of $\mathbb{P}^2$ into the a Grassmannian that they induce. Also in this case, it turns out that $d_{skew}(4,6) = 3$.

The fact that $d_{skew}(4,6) = d_{skew}(6,8) = 3$ is not surprising, since in the paper [Westwick, 1996] that we already quoted one can find the following

Lemma *In the skew-symmetric case of symmetric matrices of constant rank* r *and size* $r+2$*, if* $\dim(A) \geq 4$ *then* $r \geq 8$*.*

In fact Westwick doesn't just state (and prove) this result, he also exhibits an explicit example of such a space. Unfortunately he doesn't add any explanation on how he found his example, or on what kind of construction he used.

In the next slide you can find Westwick's mysterious example. You can try to slightly modify it to try to construct a new example, but it is quite likely that it will not work.

"The matrix

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b & 0 & c \\ 0 & 0 & 0 & 0 & 0 & -a & b & 0 & c & d \\ 0 & 0 & 0 & 0 & a & b & 0 & c & d & 0 \\ 0 & 0 & 0 & -a & 0 & 0 & c & -d & 0 & 0 \\ 0 & 0 & a & -b & 0 & 0 & d & 0 & 0 & 0 \\ 0 & -a & -b & 0 & -c & -d & 0 & 0 & 0 & 0 \\ -a & -b & 0 & -c & d & 0 & 0 & 0 & 0 & 0 \\ -b & 0 & -c & -d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c & -d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is skew symmetric, has zero determinant and is of rank ≥ 8 when any of a, b, c or d is nonzero. It therefore represents a 4-dimensional space." [Westwick 1996]

In the joint project

[A.B., D. Faenzi, and E. Mezzetti, *Linear spaces of matrices of constant rank and instanton bundles*, Advances in Mathematics 248 (2013)];

we provided an interpretation of Westwick's example in terms of instanton bundles and the derived category of $\mathbb{P}^3$, together with a construction for an infinite family of the same dimension and rank.

Later, in the other already quoted joint project

[A.B., D. Faenzi, and P.Lella, *Truncated modules and linear presentations of vector bundles*, International Mathematics Research Notices, 17 (2018)].

we gave yet another explanation in terms of graded minimal free resolutions of truncated modules.

The original construction by Westwick remains a mistery.

There are **no known examples of dimension 5**. Indeed, for $\dim(A) \geq 4$ not just the first, but also the second Chern class of the bundle K is determined by the size of the matrix:

Lemma

If A is a $(d+1)$-dimensional space of skew-symmetric matrices of constant rank r and size $r+2$, and K is the rank-2 kernel bundle, then $c_1(K) = r/2$. If moreover $d \geq 3$ then $c_2(K) = r(r+1)/12$. In particular, K is indecomposable, i.e. it cannot split as a direct sum of two line bundles.

Indecomposable rank-2 bundles on $\mathbb{P}^n$, $n \geq 4$ are notoriously hard to find. It is a famous long-standing and open conjecture that they do not exist for $n \geq 6$.

For this reason, we conjecture that $d_{skew}(r+2, r) = 4$.

A closer look at the case of order 6 and rank 4

Recall that rank 2 matrices correspond to elements of the Grassmannian $\mathbb{G}(2,6) \subseteq \mathbb{P}(\Lambda^2\mathbb{C}^6)$, and that the sth-secant variety $\sigma_s(\mathbb{G}(2,6))$ corresponds to skew-symmetric matrices of rank $r = 2s$.

Since 6×6 skew-symmetric matrices can only have rank 2, 4, or 6, there is a simple filtration of GL_6-orbits

$$\underbrace{\mathbb{G}(2,6)}_{\text{rk}=2} \quad \subseteq \quad \underbrace{\sigma_2(\mathbb{G}(2,6))}_{\text{rk}\leq 4} \quad \subseteq \quad \underbrace{\mathbb{P}(\Lambda^2\mathbb{C}^6) = \mathbb{P}^{14}}_{\text{rk}\leq 6}.$$

Notice that elements of $\mathbb{G}(2,6)$ correspond to decomposable skew-symmetric tensors, and that the locus $\sigma_2(\mathbb{G}(2,6))$ where the rank is not maximal is defined by the condition $\{\mathit{Pfaffian} = 0\}$, and is therefore a cubic hypersurface.

Therefore classifying projective planes of matrices of constant rank 4 means studying planes contained in $\sigma_2(\mathbb{G}(2,6))$ but not touching $\mathbb{G}(2,6)$.

In fact the geometry is much richer! Remark that the dual variety

$$\mathbb{G}^\vee(2,6) \simeq \sigma_2(\mathbb{G}(2,6)),$$

hence we can consider linear subspaces of $\mathbb{G}^\vee(2,6)$; via the Gauss map $\gamma : \mathbb{G}^\vee(2,6) \dashrightarrow \mathbb{G}(2,6)$ we can in fact reduce to studying linear sections of $\mathbb{G}(2,6)$.

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Linear sections of Grassmannian of lines are a classical object.

Definition

A linear section of $\mathbb{G}(2, n)$ of dimension $n - 2$ is called a linear congruence.

The classification of linear congruences in $\mathbb{P}^3$ and $\mathbb{P}^4$ is classical. For higher dimension it is still missing.

$$\boxed{\text{What does the study of linear congruences in } \mathbb{P}^5 \text{ look like?}}$$

A linear section of (the 8-dimensional variety) $\mathbb{G}(2,6)$ of dimension $6-2=4$ is of the form

$$\mathbb{G}(2,6)\cap\Delta,$$

where $\Delta\subseteq\mathbb{P}^{14}$ is a 10-dimensional linear space, intersection of four hyperplanes $H_1,H_2,H_3,H_4\in(\mathbb{P}^{14})^\vee$.

Let $\Delta^\vee=\langle H_i\rangle$ be the 3-dimensional space generated by the H_i in $(\mathbb{P}^{14})^\vee$.

Classifying linear congruences in $\mathbb{P}^5$ corresponds to describing all special positions of $\Delta^\vee$ with respect to $\mathbb{G}^\vee(2,6)$.

In general, the intersection $\Delta^\vee\cap\mathbb{G}^\vee(2,6)$ is a cubic surface; in the special case in which such cubic surface has a plane as irreducible component, this plane is exactly a **plane of skew-symmetric matrices of size 6 and constant rank 4**, which is exactly the object that we want to describe.

Connections with PDEs

The interest in this type of linear congruences is not just classical!

In the two papers

[S. Agafonov and E. Ferapontov, *Systems of conservation laws from the point of view of the projective theory of congruences*, Izv. Ross. Akad. Nauk Ser. Mat. 60 (1996)]

[S. Agafonov and E. Ferapontov, *Systems of conservation laws of Temple class, equations of associativity and linear congruences in* $\mathbb{P}^4$, Manuscripta Math. 106 (2001)]

an important connection between certain hyperbolic systems of conservation laws, called of Temple class, and congruences of lines was established.

In the more recent

[E. Ferapontov and L. Manivel, *On a class of spaces of skew-symmetric forms related to Hamiltonian systems of conservation laws*, preprint arXiv:1810.12216 (2018)]

the authors reduce the classification of n-component systems of conservation laws possessing a third-order Hamiltonian structure to the algebraic problem of classifying n-planes in $\Lambda^2(V_{n+2})$.

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Let us now see some example of 3-dimensional spaces of skew-symmetric matrices of order 6 and constant rank 4.

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Example (1)

Choose a hyperplane $\mathbb{C}^5$ in $\mathbb{C}^6$. A plane $\pi_5 \subseteq \mathbb{P}(\Lambda^2\mathbb{C}^5) \subseteq \mathbb{P}(\Lambda^2\mathbb{C}^6)$ has constant rank 4, and it is of the form

$$\pi_5 = \begin{pmatrix} 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & a & b & c & 0 \\ 0 & -a & 0 & c & 0 & 0 \\ -a & -b & -c & 0 & 0 & 0 \\ -b & -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The associated kernel bundle $K = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ splits.

Example (2)

The plane

$$\pi_g = \langle e_0 \wedge e_4 - e_1 \wedge e_3, \quad e_0 \wedge e_5 - e_2 \wedge e_3, \quad e_1 \wedge e_5 - e_2 \wedge e_4 \rangle$$

has constant rank 4 and it is of the form

$$\pi_g = \begin{pmatrix} 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & -a & 0 & c \\ 0 & 0 & 0 & -b & -c & 0 \\ 0 & a & b & 0 & 0 & 0 \\ -a & 0 & c & 0 & 0 & 0 \\ -b & -c & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Again, the associated kernel bundle $K = \mathcal{O}_{\mathbb{P}^2}(-1)^2$ splits.

Example (3)

The plane

$$\pi_t = \langle e_0 \wedge e_2 + e_1 \wedge e_3, \quad e_0 \wedge e_3 + e_1 \wedge e_4, \quad e_0 \wedge e_4 + e_1 \wedge e_5 \rangle$$

lies on the tangent space to $\mathbb{G}(2,6)$, has constant rk 4, and it is of the form

$$\pi_t = \begin{pmatrix} 0 & 0 & a & b & c & 0 \\ 0 & 0 & 0 & a & b & c \\ -a & 0 & 0 & 0 & 0 & 0 \\ -b & -a & 0 & 0 & 0 & 0 \\ -c & -b & 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 & 0 & 0 \end{pmatrix}$$

In this case the kernel bundle K doesn't split, while the image bundle is $E = \mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(1)^2$.

Example (4)

The plane

$$\pi_p = \langle e_0 \wedge e_3 + e_1 \wedge e_2, \quad e_0 \wedge e_4 + e_2 \wedge e_3, \quad e_0 \wedge e_5 + e_1 \wedge e_3 \rangle$$

has constant rank 4 and it is of the form

$$\pi_p = \begin{pmatrix} 0 & 0 & 0 & a & b & c \\ 0 & 0 & a & c & 0 & 0 \\ 0 & -a & 0 & b & 0 & 0 \\ -a & -c & -b & 0 & 0 & 0 \\ -b & 0 & 0 & 0 & 0 & 0 \\ -c & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Again, the kernel bundle K doesn't split, while the image bundle is $E = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus T_{\mathbb{P}^2}(-1)$.

It turns out that these examples illustrate all possibilities, as proved in the following

Theorem (Manivel-Mezzetti)

Any projective plane of skew-symmetric matrices of order 6 and constant rank 4 is SL_6*-equivalent to one of the four examples above.*

Theorem (Manivel-Mezzetti)

There exists no $\mathbb{P}^3$ *of skew-symmetric matrices of order 6 and constant rank 4, i.e.* $d_{skew}(4,6) = 3$.

<u>Remark.</u> In the paper [Fania-Mezzetti, 2011] a similar classification can be found for order 8 and constant rank 6.