Lecture 4

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Lecture plan (tentative!)

- Lecture 1: introduction, compression spaces, primitive spaces.
- Lecture 2: spaces of matrices of constant and bounded rank, sheaves and vector bundles; dimension bounds.
- Lecture 3: spaces of matrices and minimal free resolution of graded modules.
- **Lecture 4: the cases of symmetric and skew-symmetric matrices; applications to differential geometry and PDEs.**
- Lecture 5: applications to numerical analysis: uniform determinantal representations and compression spaces.

Symmetric and skew-symmetric matrices

Let us consider the "usual" space of matrices $A \subseteq V \otimes W$ with $\dim A = d+1$ and all non-zero elements of A have constant rank r.

We already added the hypothesis that $W = V$ (i.e. that we are dealing with square matrices), so today we exploit the decomposition

$$V \otimes V = S^2 V \oplus \Lambda^2 V,$$

and study what happens when the elements of A are either symmetric or skew-symmetric:

$$A \subseteq S^2 V, \Lambda^2 V.$$

Again, the space A gives us a long exact sequence of vector bundles on $\mathbb{P}^d$:

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^d}^n \xrightarrow{A} \mathcal{O}_{\mathbb{P}^d}(1)^n \longrightarrow N \longrightarrow 0\ ,$$

that splits into two short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & \mathcal{O}_{\mathbb{P}^d}^n \arrow[rr, "A"] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^d}(1)^n \arrow[r] & N \arrow[r] & 0\ , \\
& & & E \arrow[ur] & & &
\end{tikzcd}$$

where $n = \dim V$, and where $K = \mathrm{Ker}(A)$, $N = \mathrm{Coker}(A)$, and $E = \mathrm{Im}(A)$ are vector bundles on $\mathbb{P}^d$ of rank $n - r$, $n - r$ and r respectively.

The symmetry and skew-symmetry of the matrix A yield a symmetry of the exact sequence: indeed if we dualize everything:

$$\begin{tikzcd} 0 \arrow[r] & N^\vee \arrow[r] & \mathcal{O}_{\mathbb{P}^d}(-1)^n \arrow[rr, "A^T"] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^d}^n \arrow[r] & K^\vee \arrow[r] & 0, \\ & & & E^\vee \arrow[ur] & & & \end{tikzcd}$$

and then we tensorize by $\mathcal{O}_{\mathbb{P}^d}(1)$, we re-obtain (almost) the same sequence:

$$\begin{tikzcd} 0 \arrow[r] & N^\vee(1) \arrow[r] & \mathcal{O}_{\mathbb{P}^d}^n \arrow[rr, "A^T(1)"] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^d}(1)^n \arrow[r] & K^\vee(1) \arrow[r] & 0. \\ & & & E^\vee(1) \arrow[ur] & & & \end{tikzcd}$$

Since $A^T = A$ or $-A$, we deduce the two isomorphisms:

$$\boxed{N \simeq K^\vee(1)} \qquad \text{and} \qquad \boxed{E \simeq E^\vee(1).}$$

In particular, the two short exact sequences above now reduce to the single sequence:

$$0 \to K \to \mathcal{O}_{\mathbb{P}^d}^n \to E \to 0, \qquad \text{with} \qquad E \simeq E^\vee(1).$$

Remark that in fact for this reasoning the subspace A needs not be symmetric or skew-symmetric. For example, it could be the direct sum of symmetric and skew-symmetric subspaces.

Now recall the computation of invariants from Lecture 2: we had the equation

$$c(K)(1+t)^n = c(N),$$

that allowed us to deduce some bounds on the dimension of our spaces.

This time we can use the extra piece of information that $E \simeq E^\vee(1)$, together with the following two facts, that hold for any F vector bundle of rank r on $\mathbb{P}^d$:

1. The Chern classes of the dual bundle are $c_k(F^\vee) = (-1)^k c_k(F)$.
2. If L is a line bundle, then $c_k(F \otimes L) = \sum_{j=0}^{k} \binom{r-j}{k-j} c_j(E) c_1(L)^{k-j}$.

All in all, we obtain some linear relations on the Chern classes of E.

Let us now assume that $r = \mathrm{rk}(E) \geq d$. If $c_i(E) = e_i t^i$ then for $0 \leq i \leq d$ we have:

$$e_i = \sum_{j=0}^{i} \binom{r-j}{i-j} (-1)^j e_j.$$

For odd i, this expresses e_i as a linear combination of the e_j's with $j < i$.

In particular, if $i = 1$ and if $i = 3$ we get, respectively:

$$e_1 = \frac{r}{2} \qquad \text{and} \qquad (r-2)e_2 - 2e_3 = \frac{r(r-1)(r-2)}{12}.$$

Corollary

A space of $n \times n$ symmetric or skew-symmetric matrices of constant rank necessarily has even rank $r = 2s$. (Which is obvious for $A \subseteq \Lambda^2 V$, and surprising for $A \subseteq S^2 V$.)

If $r \leq d$ then it is possible to prove that (r is also even and)

Proposition

If $r \leq d$ then $E \simeq \mathcal{O}_{\mathbb{P}^2}^{r/2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{r/2}$ or $r = d = 2$ and $E \simeq T_{\mathbb{P}^d}(-1)$.

Remark. For $r = d = 2$ and $n = 3$ then K has rank $3 - 2 = 1$ is a line bundle and the short exact sequence is the Euler sequence on $\mathbb{P}^2$:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}^3 \to T_{\mathbb{P}^2}(-1) \to 0.$$

The symmetric case

We now wish to concentrate on the symmetric case, when $A \subseteq S^2V$. Our main references are the articles

[B.Ilic and J.M. Landsberg, *On symmetric degeneracy loci, spaces of symmetric matrices of constant rank and dual varieties*, Math. Annalen 314 (1999)]

[J.M. Landsberg, *Algebraic geometry and projective differential geometry, Seoul National University concentrated lecture series*, 1997]

As the first title suggests, the authors' interest in these spaces comes from the fact that they can be constructed from smooth varieties having degenerate duals.

Definition

Let $X \subset \mathbb{P}^N$ be a nonsingular projective variety. The dual variety $X^\vee \subset (\mathbb{P}^N)^\vee$ is the union of all hyperplanes H such that H is tangent to X, i.e. such that $H \cap X$ is singular.

A dimension count shows that one expects the dual variety $X^\vee$ to be a hypersurface in $(\mathbb{P}^N)^\vee$; it is therefore interesting to study the cases when this fails to occur.

Definition

The defect of X is $\delta := N - 1 - dim(X^\vee)$.

Example

- The smooth quadric hypersurface $Q_2 \subseteq \mathbb{P}^3$ is self dual: $Q_2^\vee \simeq Q_2$.
- The Segre variety $X = Seg(\mathbb{P}V \times \mathbb{P}W) \subset \mathbb{P}(V \otimes W)$, with $k+1 = \dim V \geq \dim W = \ell + 1$, has dual variety $\sigma_\ell(Seg(\mathbb{P}V^\vee \times \mathbb{P}W^\vee))$. Since the dimensions of these secant varieties are known, it follows that $X^\vee$ is degenerate if and only if $k > \ell$, with defect $\delta = k - \ell$.
- The Grassmannian $Gr(2, V) \subseteq \mathbb{P}(\wedge^2 V)$ of 2-dimensional subspaces in a k-dimensional vector space V has dual variety $\sigma_p(Gr(2, V^\vee))$, where $p = (k-2)/2$ if k is even and $p = (k-3)/2$ if k is odd. Again, since the dimensions of secants of Grassmannian of lines are known, it follows that $\delta = 0$ if k is even and $\delta = 2$ if k is odd.

What do spaces of symmetric matrices of constant rank have to do with varieties with degenerate duals?

Recall that in Euclidean geometry, the basic measure of how a variety is bending (that is, moving away from its embedded tangent space to first order) is the Euclidean second fundamental form.

In projective geometry, there is a projective second fundamental form that can be defined the same way as its Euclidean analogue.

For details on this, the best reference is the book

[T.A. Ivey and J.M. Landsberg, Cartan for beginners: differential geometry via moving frames and EDS, American Math. Society, 2003].

If $x \in X$ is a smooth point of our projective variety X, we denote by $II_{X,x}$ the projective second fundamental form of X at x; it generates a linear system of quadrics that we denote by $|II_{X,x}|$.

Theorem (Ilic-Landsberg)

Let $X \subseteq \mathbb{P}^N$ be a smooth variety with degenerate dual variety $X^\vee$ with defect $\delta \geq 1$, and let $H \in X^\vee$ be any smooth point. Then $|II_{X^\vee,H}|$ is a linear system of quadrics of projective dimension δ and constant rank $n - \delta$.

In other words, given a smooth variety with degenerate dual variety, this result allows to construct a $\delta + 1$ dimensional linear subspace of $S^2\mathbb{C}^{N-1-\delta}$ of constant rank $n - \delta$.

Some general remarks on this result.

- The result by [Ilic-Landsberg] can also be found in the paper

  [P. Griffiths and J. Harris, *Algebraic geometry and local differential geometry*, Ann. Scient. Ec. Norm. Sup.12(1979)],

  even though it is not stated explicitly. Indeed, they had already proved that linear spaces $A \subseteq S^2V^\vee$ of bounded rank r can be constructed from not necessarily smooth varieties which have degenerate duals.

- It also provides a new proof of a theorem of F. Zak, stating that if $X \subseteq \mathbb{P}^N$ is a non-singular projective variety, then $\dim(X^\vee) \geq \dim(X)$.

From our point of view though the main result is the following:

Theorem (Ilic-Landsberg)

If $r \geq 2$ is even, then

$$\boxed{d_{sym}(r,n) = \max\{\dim A \mid A \subseteq S^2V \text{ is of constant rank } r\} = n - r + 1.}$$

Remark that if $X_r \subseteq \mathbb{P}(S^2V)$ is the projective variety of symmetric matrices of bounded rank r, corresponding to the r-th secant variety of the Veronese variety $v_2(\mathbb{P}^{n-1})$, then an elementary dimension count shows that

$$\operatorname{codim}(X_{r-1}, X_r) = n - r + 1,$$

which explains why one expects this bound.

Finally, the result on $d(r,n)$ when r is odd is classical.

The co-rank 2 case

For both symmetric and skew-symmetric spaces of matrices, the easiest case to study is that of co-rank 2, simply because of the numerous extra-properties that rank-2 vector bundles on $\mathbb{P}^2$ satisfy.

For example, we have seen that if the space A is symmetric or skew-symmetric then the kernel bundle K and the cokernel bundle N are linked by the relation $N \simeq K^\vee(1)$.

Now, if F has rank 2, then $F^\vee \simeq F(-c_1(F))$, so that the relation above reads

$$N \simeq K(-\tfrac{r}{2} + 1).$$

In this case Westwick's bounds on the dimension read:

$$3 \leq d(r, r+2) \leq 5,$$

and in fact [Ilic-Landsberg]'s result completely solves the problem for the symmetric case, where

$$d_{sym}(r, r+2) = 3,$$

the minimal possible value.

$$\boxed{\text{What about } d_{skew}(r, r+2)?}$$