POLITECNICO DI TORINO

Lecture 3

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Lecture plan (tentative!)

- Lecture 1: introduction, compression spaces, primitive spaces.
- Lecture 2: spaces of matrices of constant and bounded rank, sheaves and vector bundles; dimension bounds.
- **Lecture 3: spaces of matrices and minimal free resolution of graded modules**
- Lecture 4: the cases of symmetric and skew-symmetric matrices; applications to differential geometry and PDEs.
- Lecture 5: applications to numerical analysis: uniform determinantal representations and compression spaces.

In this lecture, we will see how one can **use vector bundles (algebraic geometry) and truncations of graded modules (commutative algebra) to construct explicit examples of spaces of matrices of constant rank**.

This is what we did in the joint project

> [A.B., D. Faenzi, and P.Lella, *Truncated modules and linear presentations of vector bundles*, International Mathematics Research Notices, 17 (2018)].

We will also see these results "in action" in next week's tutorial with the computer software Macaulay2.

We first need some preliminary concepts from commutative algebra.

Two very good references are the books:

[I. Peeva, *Graded syzygies, Algebra and Applications*, vol. 14, Springer-Verlag London, Ltd., London, 2011],

[D. Eisenbud, *Introduction to commutative algebra with a view towards algebraic geometry*, Springer-Verlag, New York, 1995].

Let $R = \boldsymbol{k}[x_0, \ldots, x_n]$ be a homogeneous polynomial ring in $n+1$ variables over a field $\boldsymbol{k}$, $\overline{\boldsymbol{k}} = \boldsymbol{k}$, $char(\boldsymbol{k}) \neq 2$.

The ring R comes with a natural grading

$$R = \bigoplus_i R_i,$$

with $R_0 = \boldsymbol{k}$; moreover R generated by R_1 as a $\boldsymbol{k}$-algebra.

Let $\boldsymbol{M}$ be a finitely generated graded R-module, that is, $\boldsymbol{M}$ is a finitely generated R-module endowed with a direct sum decomposition $\boldsymbol{M} = \bigoplus_p \boldsymbol{M}_p$ such that $R_i \cdot \boldsymbol{M}_p \subseteq \boldsymbol{M}_{p+i}$.

If $\boldsymbol{M}$ is a finitely generated graded R-module, and $p, q \in \mathbb{Z}$, we denote by

- $\boldsymbol{M}_p$ the p-th graded component of $\boldsymbol{M}$, and by
- $\boldsymbol{M}(q)$ the q-th shift of $\boldsymbol{M}$, defined by the formula $\boldsymbol{M}(q)_p = \boldsymbol{M}_{p+q}$.
- A module $\boldsymbol{M}$ is free if $\boldsymbol{M} \simeq \bigoplus_i R(q_i)$ for suitable q_i.

Remark. Given any other finitely generated graded R-module $\boldsymbol{N}$, $\mathrm{Hom}_R(\boldsymbol{M}, \boldsymbol{N})$ is the set of homogeneous maps of all degrees, which is again a graded module, graded by the degrees of the maps.

For any R-module $\boldsymbol{M}$, we denote by $\beta_{i,j}(\boldsymbol{M})$ the graded Betti numbers of the minimal free resolution of $\boldsymbol{M}$, i.e.

$$\cdots \to \bigoplus_{j_i} R(-j_i)^{\beta_{i,j_i}} \to \cdots \to \bigoplus_{j_1} R(-j_1)^{\beta_{1,j_1}} \to \bigoplus_{j_0} R(-j_0)^{\beta_{0,j_0}} \to \boldsymbol{M} \to 0.$$

Definition

The (Castelnuovo-Mumford) regularity of a module is denoted by $\operatorname{reg}(\boldsymbol{M})$ and can be computed from the Betti numbers as $\max\{j - i \mid \beta_{i,j} \neq 0\}$.

Remark. In a Betti diagram, we see the regularity as the label of the bottom row in which we have a non-zero Betti number. Thus, the regularity is the width of the Betti diagram.

Definition

We say that $\boldsymbol{M}$ has *m-linear resolution* over R if the minimal graded free resolution of $\boldsymbol{M}$ reads:

$$\cdots \to R(-m-2)^{\beta_{2,m+2}} \to R(-m-1)^{\beta_{1,m+1}} \to R(-m)^{\beta_{0,m}} \to \boldsymbol{M} \to 0$$

for suitable integers $\beta_{i,m+i}$.

In the case where only the first k maps are matrices of linear forms then $\boldsymbol{M}$ is called *m-linearly presented up to order k*, or just *linearly presented* when $k = 1$.

In other words, $\boldsymbol{M}$ has a m-linear resolution if:

1. $\boldsymbol{M}_r = 0$ for $r < m$,
2. $\boldsymbol{M}$ is generated by $\boldsymbol{M}_m$, and
3. $\boldsymbol{M}$ has a resolution where all maps are matrices of **linear forms**.

Definition

The module

$$\boldsymbol{M}_{\geq m} = \bigoplus_{p \geq m} \boldsymbol{M}_p$$

is the truncation of $\boldsymbol{M}$ at degree m.

Example

Let $T = (a^3, b^3, c^3, ab, ac)$ in the ring $R = \boldsymbol{k}[a, b, c]$. We have that

$$T_{\geq 3} = (a^3, b^3, c^3, abc, a^2b, ab^2, a^2c, ac^2).$$

Theorem

The truncation $\boldsymbol{M}_{\geq \mathrm{reg}(\boldsymbol{M})}$ always has m-linear resolution.

Recall that we are looking for a way to construct linear matrices starting from a vector bundle.

Let E be a vector bundle on $\mathbb{P}^n = Proj(R), \quad R = \mathbb{C}[x_0, \dots, x_n]$.

<u>**First idea:**</u> look at its minimal free resolution. Even better, resolve its module of sections:

$$\boldsymbol{E} = \mathrm{H}^0_*(E) = \bigoplus_{t \in \mathbb{Z}} \mathrm{H}^0(E(t)),$$

a graded R-module with graded Betti numbers $\beta_{i,j}$ and minimal free resolution:

$$\cdots \to \bigoplus_{j_1} R(-j_1)^{\beta_{1,j_1}} \to \bigoplus_{j_0} R(-j_0)^{\beta_{0,j_0}} \to \boldsymbol{E}.$$

This is of course **too naïve**. The resolution is in general **not** linear!

We can get around the non-linearity of the resolution of $\boldsymbol{E}$ by **truncating the graded module "in the right spot"**, namely its regularity.

Indeed, we saw that if $\boldsymbol{E} = \bigoplus_{t \in \mathbb{Z}} \boldsymbol{E}_t$, then $\boldsymbol{E}_{\geq m} = \bigoplus_{t \geq \mathbb{Z}} \boldsymbol{E}_t$ and $\boldsymbol{E}_{\geq \mathrm{reg}(\boldsymbol{E})}$ always has m-linear resolution.

So we obtained linearity, but we still need something a bit more sophisticated! To begin with, we **lost control over the size** of the matrix.

(Vague) idea: "cut off a piece" of the linear matrix from the linear resolution of $\boldsymbol{E}_{\geq \mathrm{reg}(\boldsymbol{E})}$, **without modifying the rank**.

Let $\boldsymbol{E}$ and $\boldsymbol{G}$ be f.g. graded R-modules with minimal graded free resolutions:

$$\cdots \longrightarrow E^1 \xrightarrow{e_1} E^0 \xrightarrow{e_0} \boldsymbol{E}$$

$$\cdots \longrightarrow G^1 \xrightarrow[g_1]{} G^0 \xrightarrow[g_0]{} \boldsymbol{G}$$

Let $\boldsymbol{E}$ and $\boldsymbol{G}$ be f.g. graded R-modules with minimal graded free resolutions:

A morphism $\mu : \boldsymbol{E} \to \boldsymbol{G}$ induces maps $\mu^i : E^i \to G^i$, determined up to chain homotopy.

(If $\boldsymbol{E}$ and $\boldsymbol{G}$ are linearly presented up to order j, then the μ^i's are uniquely determined for $i \leq j - 1$.)

$$\begin{array}{ccccccc}
\cdots \longrightarrow & E^1 & \xrightarrow{e_1} & E^0 & \xrightarrow{e_0} & \boldsymbol{E} \\
 & \downarrow \mu^1 & & \downarrow \mu^0 & & \downarrow \mu \\
\cdots \longrightarrow & G^1 & \xrightarrow[g_1]{} & G^0 & \xrightarrow[g_0]{} & \boldsymbol{G}
\end{array}$$

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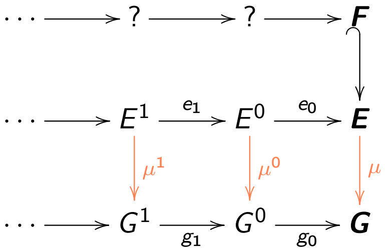

What can we say about the resolution of the kernel $\boldsymbol{F}$?

Theorem

Let $\boldsymbol{E}$ and $\boldsymbol{G}$ be two m-linearly presented R-modules, respectively up to order 1 and 2. Suppose that there is a surjective morphism $\mu : \boldsymbol{E} \twoheadrightarrow \boldsymbol{G}$, and let μ^i's be the induced maps.

Then $\boldsymbol{F} = \ker(\mu)$ is generated in deg m and $m+1$, and moreover:

1. *if μ^1 is surjective, $\boldsymbol{F}$ is generated in deg m and has linear and quadratic syzygies, and $\beta_{0,m}(\boldsymbol{F}) = \beta_{0,m}(\boldsymbol{E}) - \beta_{0,m}(\boldsymbol{G})$;*
2. *if moreover μ^2 is surjective, $\boldsymbol{F}$ is linearly presented and $\beta_{1,m+1}(\boldsymbol{F}) = \beta_{1,m+1}(\boldsymbol{E}) - \beta_{1,m+1}(\boldsymbol{G})$.*

Theorem (revisited)

Let $\boldsymbol{E}$ and $\boldsymbol{G}$ be two m-linearly presented R-modules, respectively up to order 1 and 2. Suppose that there is a surjective morphism $\mu : \boldsymbol{E} \twoheadrightarrow \boldsymbol{G}$, and let μ^i's be the induced maps.

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2.

$$
\begin{array}{ccccccccc}
 & & \cdots & \longrightarrow & R(-m-1)^{\alpha_1-\gamma_1} & \longrightarrow & R(-m)^{\alpha_0-\gamma_0} & \longrightarrow & \boldsymbol{F} \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow & R(-m-2)^{\alpha_2} & & \longrightarrow & R(-m-1)^{\alpha_1} & \longrightarrow & R(-m)^{\alpha_0} & \longrightarrow & \boldsymbol{E} \\
 & \downarrow \mu^2 & & & \downarrow \mu^1 & & \downarrow \mu^0 & & \downarrow \mu \\
\cdots \longrightarrow & R(-m-2)^{\gamma_2} & & \longrightarrow & R(-m-1)^{\gamma_1} & \longrightarrow & R(-m)^{\gamma_0} & \longrightarrow & \boldsymbol{G}
\end{array}
$$

Remember once again that our goal is to construct constant rank matrices!
What happens if the sheaves $E = \tilde{\boldsymbol{E}}$ and $G = \tilde{\boldsymbol{G}}$ are vector bundles?

Theorem

In the assumptions of the previous Theorem, part 1, suppose also that:

(i) *$E = \tilde{\boldsymbol{E}}$ and $G = \tilde{\boldsymbol{G}}$ are v.b. on $\mathbb{P}^n$ of rank r and s respectively;*

(ii) *some extra "technical condition" holds.*

Set $a = \beta_{0,m}(\boldsymbol{E}) - \beta_{0,m}(\boldsymbol{G})$ and $b = \beta_{1,m+1}(\boldsymbol{E}) - \beta_{1,m+1}(\boldsymbol{G})$.

Then the presentation matrix of $\boldsymbol{F} = \ker(\mu)$ has a linear part of size $a \times b$ and constant corank $r - s$.

Moreover $F = \tilde{\boldsymbol{F}}$ is isomorphic to the kernel of $\tilde{\mu} : E \to G$.

Remark. μ^2 surjective $\underset{\not\Leftarrow}{\Rightarrow}$ technical condition.

In the applications we consider the case when $G = 0$, i.e. $\boldsymbol{G}$ Artinian module, so in particular $F = \tilde{\boldsymbol{F}} \simeq \tilde{\boldsymbol{E}} = E$.

From this construction we get a veritable factory of examples of constant rank matrices!

Moreover, we can implement the method on a computer (using the package ConstantRankMatrices implemented using Macaulay2 software), and avoid cumbersome computations.

You can find the package at `http://www.paololella.it/IT/Software.html`, and we will look at it together in next week's tutorial.