Lecture 2

Ada Boralevi

Third Research Schools on Commutative Algebra and Algebraic Geometry, Applied Algebraic Geometry@ IASBS, Zanjan
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Lecture plan (tentative!)

- Lecture 1: introduction, compression spaces, primitive spaces.
- **Lecture 2: spaces of matrices of constant and bounded rank, sheaves and vector bundles; dimension bounds.**
- Lectures 3 & 4: the cases of symmetric and skew-symmetric matrices; applications to differential geometry and PDEs.
- Lecture 5: applications to numerical analysis: uniform determinantal representations and compression spaces.

Recall from the previous lecture:

Given two complex vector spaces V and W of dimension m and n respectively, a space of matrices of constant rank is a $d+1$-dimensional vector subspace $A \subseteq V^\vee \otimes W \simeq \mathrm{Hom}(V, W)$ whose non-zero elements all have the same rank r.

The inclusion $A \hookrightarrow V^\vee \otimes W$ is an element of $A^\vee \otimes V^\vee \otimes W$; if we identify $A^\vee \simeq \mathrm{H}^0(\mathcal{O}_{\mathbb{P}A}(1))$, then we obtain an element of

$$\mathrm{Hom}\left(V \otimes \mathcal{O}_{\mathbb{P}A}, W \otimes \mathcal{O}_{\mathbb{P}A}(1)\right),$$

whose rank is constant. Therefore we can **think of A as an m $\times$ n matrix whose entries are linear forms in d + 1 variables**, i.e. a vector bundle map

$$\phi : V \otimes \mathcal{O}_{\mathbb{P}A} \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}A}(1)$$

that, evaluated at every point of $\mathbb{P}A$, has the same rank r.

Even more in detail, to A we can associate a vector bundle map

$$\psi : V \otimes \mathcal{O}_{\mathbb{P}A}(-1) \to W \otimes \mathcal{O}_{\mathbb{P}A}$$

on $\mathbb{P}A$ as follows: at $[q] \in \mathbb{P}A$ the fiber of $\mathcal{O}_{\mathbb{P}A}(-1)$ is λq, $\lambda \in \mathbb{C}$ and we set

$$\psi(v \otimes \lambda q) = \lambda \cdot q(v).$$

Tensoring by $\mathcal{O}_{\mathbb{P}A}(1)$ we get the map ϕ described above.

⚠ In fact ϕ carries all the information of the space A: taking global sections we can reverse the construction and get the map $V \to W \otimes A^\vee$, adjoint to the inclusion $A \hookrightarrow V^\vee \otimes W$.

However you want to look at it, we obtain a long exact sequence

$$\boxed{0 \longrightarrow K_A \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}A} \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}A}(1) \longrightarrow N_A \longrightarrow 0,}$$

where K_A and N_A denote the kernel and cokernel of A.

We also consider the image E_A of the map A, splitting the long exact sequence into two short exact sequences, we get:

$$\begin{tikzcd}
0 \arrow[r] & K_A \arrow[r] & V \otimes \mathcal{O}_{\mathbb{P}A} \arrow[rr] \arrow[dr] & & W \otimes \mathcal{O}_{\mathbb{P}A}(1) \arrow[r] & N_A \arrow[r] & 0 . \\
& & & E_A \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

The assumption that all elements of A have constant rank r implies that the image, the kernel and cokernel can be interpreted as vector spaces varying smoothly over $\mathbb{P}^d$, i.e. vector bundles, whose rank is as follows:

$$\mathrm{rk}(E_A) = r, \qquad \mathrm{rk}(K_A) = m - r, \qquad \mathrm{rk}(N_A) = n - r.$$

We are now ready to continue the discussion on Eisenbud-Harris' paper and see how the geometry of these vector bundles relates to the space A.

Remark. You may have noticed that we are not taking into account the bounded rank case anymore; the reason is that we would need to consider the more general concept of (torsion free) sheaves, which goes beyond the scope of these lectures. **From now on, we will only discuss the case of constant rank**.

Theorem (Eisenbud-Harris)

Let $\phi : V \otimes \mathcal{O}_{\mathbb{P}A} \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}A}(1)$ *be the vector bundle map associated to a space of matrices of constant rank* r $A \subset \mathrm{Hom}(V, W)$.
Let $E_A = \mathrm{Im}(\phi)$ *and define* $F_A = \mathrm{Im}\left(\phi^\vee(1) : W^\vee \otimes \mathcal{O}_{\mathbb{P}A} \to V^\vee \otimes \mathcal{O}_{\mathbb{P}A}(1)\right)$.

The following conditions are equivalent:

(i) *A is a compression space;*

(ii) *E_A is a direct sum of rank 1 bundles;*

(iii) *E_A and F_A have as direct summands trivial vector bundles of ranks r_1 and r_2, with $r = r_1 + r_2$.*

Moreover the summands in (ii) are necessarily copies of $\mathcal{O}_{\mathbb{P}A}$ and $\mathcal{O}_{\mathbb{P}A}(1)$.

Recall from our tutorial:

Theorem (Segre-Grothendieck)

Every rank r vector bundle E on $\mathbb{P}^1$ splits as a direct sum $E = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some integers a_i, not necessarily distinct.

Combining Segre-Grothendieck with last slide we get a proof of the following (classical) result:

Characterization of matrix pencils

Every 2-dimensional space of matrices of constant rank is a compression space.

There is a similar characterization of primitivity:

Theorem (Eisenbud-Harris)

With the same hypotheses as above, the following conditions are equivalent:

(i) *A is a not primitive;*

(ii) *E_A or F_A has a rank 1 summand.*

Let us now move on to see more ways in which vector bundles can help us in the study of spaces of matrices.

We saw that, while 2-dimensional spaces of matrices are completely classified, things get more complicated as soon as $\dim(A) \geq 3$.

Naturally the question arises, **how "big" can these spaces get?**

Definition

Given V and W two complex vector spaces of dimension m and n respectively, we define:

$$\boxed{d(r, m, n) = \max \left\{ \dim(A) \mid A \subset \mathrm{Hom}(V, W) \text{ has constant rank } r \right\}.}$$

The following construction, due to J. Sylvester in

[*On the dimension of spaces of linear transformations satisfying rank conditions*, Linear Algebra and its Applications 78 (1986)],

and implemented by R. Westwick in

[*Spaces of matrices of fixed rank*, Linear and Multilinear Algebra 20 (1987)],

and later in

[*Spaces of matrices of fixed rank, II*, Linear Algebra and its Applications 235 (1996)],

allows one to establish some bounds on $d(r, m, n)$.

Recall the definition of Chern classes.

If F is any vector bundle on $\mathbb{P}^d$, we can write $c_i(F) = f_i t^i$ for some integer $f_i \in \mathbb{Z}$, where $t = c_1(\mathcal{O}_{\mathbb{P}^d}(1))$.

Its Chern polynomial is

$$c(F) = 1 + c_1(F) + \ldots + c_d(F) = 1 + f_1 t + \ldots + f_d t^d.$$

If $0 \to F_1 \to F \to F_2 \to 0$ is a short exact sequence of vector bundles, then

$$c(F) = c(F_1)c(F_2).$$

Consider again the two short exact sequences induced by the map

$$\phi : \mathcal{O}_{\mathbb{P}^d}^m \to \mathcal{O}_{\mathbb{P}^d}(1)^n :$$

$$\textcolor{red}{\bigstar} \qquad 0 \to K_A \to \mathcal{O}_{\mathbb{P}^d}^m \to E_A \to 0$$

$$\textcolor{blue}{\bigstar} \qquad 0 \to E_A \to \mathcal{O}_{\mathbb{P}^d}(1)^n \to N_A \to 0$$

with $E_A = \mathrm{Im}(\phi)$, $K_A = \mathrm{Ker}(\phi)$, and $N_A = \mathrm{Coker}(\phi)$.

From $\textcolor{red}{\bigstar}$ we get $c(K_A)c(E_A) = 1$, and from $\textcolor{blue}{\bigstar}$ we get $c(E_A)c(N_A) = (1+t)^n$, therefore

$$c(K_A)(1+t)^n = c(N_A).$$

Since $\mathrm{rk}(N_A) = n - r$, if $n - r + 1 \leq i \leq d$, then $c_i(N_A) = 0$ and looking at the coefficient of t^i we get $\sum_{j=0}^{m-r} \binom{n}{i-j} c_j(K_A) = 0$.

Suppose that $d = m + n - 2r + 1$; then the coefficient matrix of this linear system is square with non-zero determinant, and this forces $c_0(K_A) = 0$, a contradiction. This proves the first part of the following result.

Theorem (Westwick)

Suppose $2 \leq r \leq m \leq n$. Then

(1) $d(r, m, n) \leq m + n - 2r + 1$;

(2) $d(r, m, n) = n - r + 1$ *if* $n - r + 1$ *does not divide* $(m-1)!/(r-1)!$;

(3) $d(r, r+1, 2r-1) = r + 1$.

The theorem above leaves unanswered the case $d(rs, rs+1, rs+s-1)$, that could be either s or $s+1$. Westwick himself proves that it is the latter, in the paper

[*Examples of constant rank spaces*, Linear and Multilinear Algebra 28 (1990)].

To do so, he provides the following explicit construction: let $x_0, x_1, \dots$ be an infinite list of independent variables. For each $r \geq 1$, define a matrix $H_r = (h_{ij})$ whose entries are linear monomials in the x_i as follows:

$$h_{i,j} = \begin{cases} 0 & \text{if} \quad i-j \geq 2, \\ x_t & \text{if} \quad i-j = 1-t \quad \text{and } j \not\equiv 0 \bmod (r+1), \\ (a-t)x_t & \text{if} \quad i-j = 1-t \quad \text{and } j = a(r+1). \end{cases}$$

Let $H_{r,s}$ to be the leading $(rs+1) \times (rs+s-1)$ submatrix of H_r with $x_k = 0$ for $k \geq s+1$. For example, when $r = 2$ and $s = 3$:

$$H_{2,3} = \begin{pmatrix} x_1 & x_2 & -2x_3 & x_4 & x_5 & -4x_6 & x_7 & x_8 \\ x_0 & x_1 & -x_2 & x_3 & x_4 & -3x_5 & x_6 & x_7 \\ & x_0 & 0 & x_2 & x_3 & -2x_4 & x_5 & x_6 \\ & & x_0 & x_1 & x_2 & -x_3 & x_4 & x_5 \\ & & & x_0 & x_1 & 0 & x_3 & x_4 \\ & & & & x_0 & x_1 & x_2 & x_3 \\ & & & & & 2x_0 & x_1 & x_2 \end{pmatrix}$$

It easy to see that $\mathrm{rk}(H_{N,r}) \geq rN$; to prove the opposite inequality one needs a very complicated construction of an appropriate annihilator.

In the joint project

[A.B., D. Faenzi, and P. Lella, *A construction of equivariant bundles on the space of symmetric forms*, arXiv:1804.06211 (2018)]

we take a completely different approach and introduce a new technique to construct non-splitting vector bundles on $\mathbb{P}^N$, looking at it as the space of homogeneous forms of degree d on $\mathbb{P}^n$ for some (d, n).

By definition their dual bundles are presented by a matrix of linear forms which is also equivariant for the action of $\mathrm{SL}_{n+1}(\mathbb{C})$. When $n = 1$ (and thus $N = d$), the matrix has the correct size and rank to achieve Westwick's bound, but with a much simpler construction.

The construction can be implemented in Macaulay2 and we developed the package SLnEquivariantMatrices, that can be found at `http://www.paololella.it/IT/Software.html`.

In general the effective value of $d(r, m, n)$ is unknown, and the bounds we have are far from being sharp. We then impose some extra constraints, hoping to get better results.

For example, if we assume $V = W$, we only need to deal with square matrices. There are several partial results on the value of $d(r, n) = d(r, n, n)$.

From the paper

[L.B. Beasley, *Spaces of matrices of equal rank*, Linear Algebra and its Applications, 38 (1981)]

we learn that:

- $d(r, n) \leq \max\{r + 1, n - r + 1\}$;
- if $n \geq 2r$, then $d(r, n) = n - r + 1$;
- $d(r + 1, 2r + 1) = r + 2$.

In the already quoted papers by Sylvester and Westwick the following bounds are shown:

- [Sylvester, 1986]: $d(n-1,n) = \begin{cases} 2, & \text{if n is even} \\ 3, & \text{if n is odd} \end{cases}$
- [Westwick, 1996]: $d(r,n) \leq 2n - 2r + 1$
- [Westwick, 1987]: $3 \leq d(n-2,n) \leq 5$, moreover
    - $d(n-2,n) \leq 4$, except if $a \equiv 2, 10 \pmod{12}$, where $d(n-2,n)$ could be 5;
    - if $a \equiv 0 \pmod 3$, then $d(n-2,n) = 3$;
    - if $a \equiv 1 \pmod 3$, then $d(2,4) = 3$ and $d(8,10) = 4$, hence n doesn't determine $d(n-2,n)$;
    - if $a \equiv 2 \pmod 3$, then $d(n-2,n) \geq 4$, and if $n \not\equiv 2 \pmod 4$ then $d(n-2,n) = 4$.

There are other tools from algebraic geometry that one can exploit, such as properties of uniform vector bundles: this is done in

[Ph. Ellia and P. Menegatti, *Spaces of matrices of constant rank and uniform vector bundles*, Linear Algebra and its Applications 507 (2016)].

Recall once again that by Segre-Grothendieck, if we restrict a vector bundle to a line, it splits as a direct sum of line bundles.

Definition

A rank r vector bundle E on $\mathbb{P}^d$ is uniform if there exists $(a_1, \ldots, a_r)$ such that $E_{|_L} \simeq \oplus_{i_1}^r \mathcal{O}_L(a_i)$ for every line $L \subset \mathbb{P}^d$.

Hence "uniform" means that the splitting type is independent of the line.

Let's look again at the image bundle $E = E_A$ of the map A, from the sequence:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & \mathcal{O}_{\mathbb{P}^d}^n \arrow[rr] \arrow[dr, two heads] & & \mathcal{O}_{\mathbb{P}^d}^n(1) \arrow[r] & N \arrow[r] & 0 . \\ & & & E \arrow[ur, hook] & & & \end{tikzcd}$$

Consider the restriction of E to any line $L \subset \mathbb{P}^d$: $E_{|_L} = \oplus_{i=1}^r \mathcal{O}_L(a_i)$.

Since $E_{|_L} \hookrightarrow \mathcal{O}_L(1)^n$, we must have $a_i \leq 1$; on the other hand, since $\mathcal{O}_L^n \twoheadrightarrow E$, we must have $a_i \geq 0$. So $0 \leq a_i \leq 1$, for all i.

Moreover $\sum_{i=1}^r a_i = c_1(E) = -c_1(K)$, hence the splitting type does not depend on the line L, meaning that E is a **uniform vector bundle**.

Using some classical results on uniform bundles, Ellia-Menegatti prove the following result:

Theorem

1. *If* $r \leq n/2$, *then* $d(r,n) = d - r + 1$;
2. *if* n *is odd,* $d(\frac{n+1}{2}, n) = \frac{n+1}{2} + 1 (= n - r + 2)$;
3. *if* $\frac{2n+2}{3} > r \leq n/2 + 1$, *then* $d(r,n) \leq r - 1$;
4. *if* n *is even:* $d(n/2+1, n) = n/2 (= n - r + a)$;
5. *if* $r \geq \frac{2n+2}{3}$, *then* $d(r,n) \leq 2n - 2r + 1$;
6. $d(5,7) = 3 (= n - r + 1)$.

In view of these results, and of a famous longstanding conjecture regarding uniform bundles, they make the following

Conjecture

Let n, r be integers such that $\frac{2n+2}{3} > r > \frac{a}{2} + 1$, then

$$d(r, n) = n - r + 1.$$

All the partial results listed above imply that the conjecture is **true for n** $\leq$ **10**.

⚠ It should be apparent by now that the problem of finding the effective value of $d(r, m, n)$ is still open and far from easy!